CHAPTER 1

Some Correlation Basics: Properties, Motivation, Terminology

Behold, the fool saith, "Put not all thine eggs in the one basket."
—Mark Twain

In this chapter we introduce the basic concepts of financial correlations and financial correlation risk. We show that correlations are critical in many areas of finance such as investments, trading, and risk management, as well as in financial crises and in financial regulation. We also show how correlation risk relates to other risks in finance such as market risk, credit risk, systemic risk, and concentration risk.

1.1 WHAT ARE FINANCIAL CORRELATIONS?

Heuristically (meaning nonmathematically), we can define two types of financial correlations: static and dynamic.

Static financial correlations measure how two or more financial assets are associated within a certain time period.

Examples are:

- The classic value at risk (VaR) model. It answers the question: What is the maximum loss of correlated assets in a portfolio with a certain probability for a given time period? This time period can be 10 days as Basel III requires, as well as shorter or longer (see Chapter 1, section 1.3.3 and Chapter 9, section 9.4 for more on VaR and correlation).

- The original copula approach for collateralized debt obligations (CDOs). It measures the default correlations between all assets in the CDO for a certain time period, which is typically identical to the maturity date of the CDO (see Chapter 5 for details).
- The binomial default correlation model of Lucas (1995), which is a special case of the Pearson correlation model. It measures the probability of two assets defaulting together within a short time period (see Chapter 3 for details).

Besides the static correlation concept, there are dynamic correlations:

Dynamic financial correlations measure how two or more financial assets move together in time.

Examples are:

- In practice, pairs trading, a type of statistical arbitrage, is performed. Let's assume the movements of assets x and y have been highly correlated in time. If now asset x performs poorly with respect to y, then asset x is bought and asset y is sold with the expectation that the gap will narrow.
- Within the deterministic correlation approaches, the Heston 1993 model correlates the Brownian motions dz_1 and dz_2 of assets 1 and 2. The core equation is $dz_1(t) = \rho\, dz_2(t) + \sqrt{(1 - p^2)}\, dz_3(t)$ where dz_1 and dz_2 are correlated in time with correlation parameter p. See Chapter 3 for details.
- Correlations behave randomly and unpredictably. Therefore, it is a good idea to model them as a stochastic process. Stochastic correlation processes are by construction time dependent and can replicate correlation properties well. See Chapter 8 for details.

> *Suddenly everything was highly correlated.*
>
> —*Financial Times*, April 2009

1.2 WHAT IS FINANCIAL CORRELATION RISK?

Financial correlation risk is the risk of financial loss due to adverse movements in correlation between two or more variables.

These variables can comprise any financial variables. For example, the positive correlation between Mexican bonds and Greek bonds can hurt

Mexican bond investors if Greece bond prices decrease, which happened in 2012 during the Greek crisis. Or the negative correlation between commodity prices and interest rates can hurt commodity investors if interest rates rise. A further example is the correlation between a bond issuer and a bond insurer, which can hurt the bond investor (see the example displayed in Figure 1.1).

Correlation risk is especially critical in risk management. An increase in the correlation of asset returns increases the risk of financial loss, which is often measured by the value at risk (VaR) concept. For details see section 1.3.3 and Chapter 9, sections 9.4 and 9.5. An increase in correlation is typical in a severe systemic crisis. For example, in the Great Recession from 2007 to 2009, financial assets and financial markets worldwide became highly correlated. Risk managers who had in their portfolios assets with negative or low correlations suddenly witnessed many of them decline together; hence asset correlations increased sharply. For more on systemic risk, see section 1.3.4, "The Global Financial Crisis of 2007 to 2009 and Correlation," as well as Chapter 2, which displays empirical findings of correlations.

Correlation risk can also involve variables that are nonfinancial, such as economic or political events. For example, the correlation between the increasing sovereign debt and currency value can hurt an exporter, as occurred in Europe in 2012, where a decreasing euro hurt U.S. exporters. Geopolitical tensions as, for example, in the Middle East can hurt airline companies due to increasing oil prices, or a slowing gross domestic product (GDP) in the United States can hurt Asian and European exporters and investors, since economies and financial markets are correlated worldwide.

Let's look at correlation risk via an example of a credit default swap (CDS). A CDS is a financial product in which the credit risk is transferred from the investor (or CDS buyer) to a counterparty (CDS seller). Let's assume an investor has invested $1 million in a bond from Spain. He is now worried about Spain defaulting and has purchased a credit default swap from a French bank, BNP Paribas. Graphically this is displayed in Figure 1.1.

The investor is protected against a default from Spain, since in case of default, the counterparty BNP Paribas will pay the originally invested $1 million to the investor. For simplicity, let's assume the recovery rate and accrued interest are zero.

The value of the CDS, i.e., the fixed CDS spread s, is mainly determined by the default probability of the reference entity Spain. However, the spread s is also determined by the joint default correlation of BNP Paribas and Spain. If the correlation between Spain and BNP Paribas increases, the present value of the CDS for the investor will decrease and he will suffer a paper loss. The worst-case scenario is the joint default of Spain and BNP Paribas, in

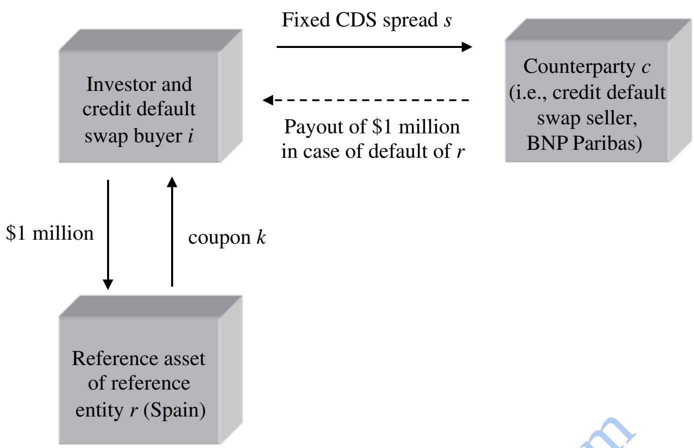

FIGURE 1.1 An Investor Hedging His Spanish Bond Exposure with a CDS

which case the investor will lose his entire investment in the Spanish bond of $1 million.

In other words, the investor is exposed to default correlation risk between the reference asset r (Spain) and the counterparty c (BNP Paribas). Since both Spain and BNP Paribas are in Europe, let's assume that there is a positive default correlation between the two. In this case, the investor has wrong-way correlation risk or short wrong-way risk (WWR). Let's assume the default probability of Spain and BNP Paribas both increase. This means that the exposure to the reference entity Spain increases (since the CDS has a higher present value for the investor) *and* it is more unlikely that the counterparty BNP Paribas can pay the default insurance. We will discuss wrong-way risk, which is a key term in the Basel II and III accords, in Chapter 12.

The magnitude of the correlation risk is expressed graphically in Figure 1.2.

From Figure 1.2 we observe that for a correlation of $-0{,}3$ and higher, the higher the correlation, the lower the CDS spread. This is because an increasing ρ means a higher probability of the reference asset and the counterparty defaulting together. In the extreme case of a perfect correlation of 1, the CDS is worthless. This is because if Spain defaults, so will the insurance seller BNP Paribas.

We also observe from Figure 1.2 that for a correlation from -1 to about -0.3, the CDS spread increases slightly. This seems counterintuitive at first. However, an increase in the negative correlation means a higher probability of either Spain *or* BNP Paribas defaulting. In the case of Spain defaulting, the CDS buyer will get compensated by BNP Paribas. However, if the insurance seller BNP Paribas defaults, the CDS buyer will lose his

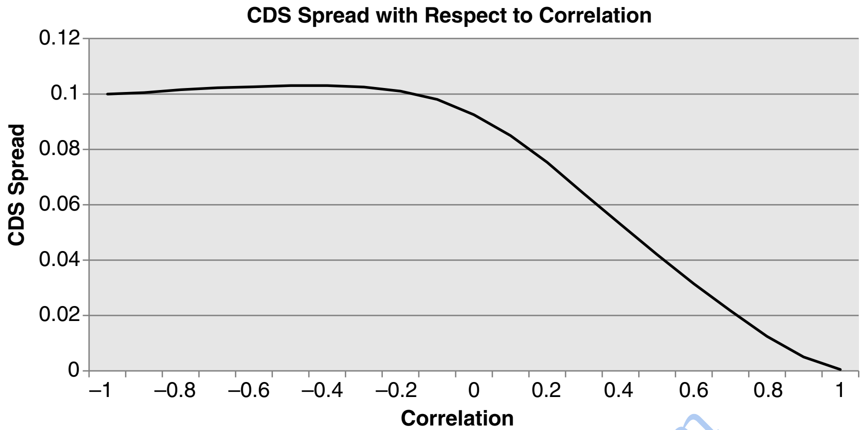

FIGURE 1.2 CDS Spread *s* of a Hedged Bond Purchase (as Displayed in Figure 1.1) with Respect to the Default Correlation between the Reference Entity *r* and the Counterparty *c*

insurance and will have to repurchase it. This may have to be done at a higher cost. The cost will be higher if the credit quality of Spain has decreased since inception of the original CDS. For example, the CDS spread may have been 3% in the original CDS, but may have increased to 6% due to a credit deterioration of Spain. For more details on pricing CDSs with counterparty risk and the reference asset–counterparty correlation, see Chapter 10, section 10.1, as well as Kettunen and Meissner (2006).

We observe from Figure 1.2 that the dependencies between a variable (here the CDS spread) and correlation may be nonmonotonous; that is, the CDS spread sometimes increases and sometimes decreases if correlation increases. We will also encounter this nonmonotony feature of correlation when we discuss the mezzanine tranche of a CDO in Chapter 5.

1.3 MOTIVATION: CORRELATIONS AND CORRELATION RISK ARE EVERYWHERE IN FINANCE

Why study financial correlations? That's an easy one. Financial correlations appear in many areas in finance. We will briefly discuss five areas: (1) investments and correlation, (2) trading and correlation, (3) risk management and correlation, (4) the global financial crisis and correlation, and (5) regulation and correlation. Naturally, if an entity is exposed to correlation, this means that the entity has correlation risk (i.e., the risk of a change in the correlation).

1.3.1 Investments and Correlation

From our studies of the Nobel Prize–winning capital asset pricing model (CAPM) (Markowitz 1952; Sharpe 1964) we remember that an increase in diversification increases the return/risk ratio. Importantly, high diversification is related to low correlation. Let's show this in an example. Let's assume we have a portfolio of two assets, X and Y. They have performed as in Table 1.1.

Let's define the return of asset X at time t as x_t, and the return of asset Y at time t as y_t. A return is calculated as a percentage change, $(S_t - S_{t-1})/S_{t-1}$, where S is a price or a rate. The average return of asset X for the time frame 2009 to 2013 is $\mu_X = 29.03\%$; for asset Y the average return is $\mu_Y = 20.07\%$. If we assign a weight to asset X, w_X, and a weight to asset Y, w_Y, the portfolio return is

$$\mu_P = w_X \mu_X + w_Y \mu_Y \tag{1.1}$$

where $w_X + w_Y = 1$.

The standard deviation of returns, called *volatility*, is derived for asset X with equation (1.2):

$$\sigma_X = \sqrt{\frac{1}{n-1}\sum_{t=1}^{n}(x_t - \mu_X)^2} \tag{1.2}$$

where x_t is the return of asset X at time t and n is the number of observed points in time. The volatility of asset Y is derived accordingly. Equation 1.2 can be computed with =stdev in Excel and std in MATLAB. From our example in Table 1.1, we find that $\sigma_X = 44.51\%$ and $\sigma_Y = 47.58\%$.

Let's now look at the covariance. The covariance measures how two variables covary (i.e., move together). More precisely, the covariance

TABLE 1.1 Performance of a Portfolio with Two Assets

Year	Asset X	Asset Y	Return of Asset X	Return of Asset Y
2008	100	200		
2009	120	230	20.00%	15.00%
2010	108	460	−10.00%	100.00%
2011	190	410	75.93%	−10.87%
2012	160	480	−15.79%	17.07%
2013	280	380	75.00%	−20.83%
		Average	29.03%	20.07%

measures the strength of the linear relationship between two variables. The covariance of returns for assets X and Y is derived with equation (1.3):

$$\mathrm{Cov}_{XY} = \frac{1}{n-1}\sum_{t=1}^{n}(x_t - \mu_X)(y_t - \mu_Y) \tag{1.3}$$

For our example in Table 1.1 we derive $\mathrm{Cov}_{XY} = -0.1567$. Equation (1.3) is = Covariance.S in Excel and cov in MATLAB. The covariance is not easy to interpret, since it takes values between $-\infty$ and $+\infty$. Therefore, it is more convenient to use the Pearson correlation coefficient ρ_{XY}, which is a standardized covariance; that is, it takes values between -1 and $+1$. The Pearson correlation coefficient is:

$$\rho_{XY} = \frac{\mathrm{Cov}_{XY}}{\sigma_X \sigma_Y} \tag{1.4}$$

For our example in Table 1.1, $\rho_{XY} = -0.7403$, showing that the returns of assets X and Y are highly negatively correlated. Equation (1.4) is 'correl' in Excel and 'corrcoef' in MATLAB. For the derivation of the numerical examples of equations (1.2) to (1.4) and more information on the covariances, see Appendix 1A and the spreadsheet "Matrix primer.xlsx," sheet "Covariance matrix," at www.wiley.com/go/correlationriskmodeling under "Chapter 1."

We can calculate the standard deviation for our two-asset portfolio P as

$$\sigma_P = \sqrt{w_X^2\sigma_X^2 + w_Y^2\sigma_Y^2 + 2w_X w_Y \mathrm{Cov}_{XY}} \tag{1.5}$$

With equal weights, i.e., $w_X = w_Y = 0.5$, the example in Table 1.1 results in $\sigma_P = 16.66\%$.

Importantly, the standard deviation (or its square, the variance) is interpreted in finance as risk. The higher the standard deviation, the higher the risk of an asset or a portfolio. Is standard deviation a good measure of risk? The answer is: It's not great, but it's one of the best we have. A high standard deviation may mean high upside potential, so it penalizes possible profits! But a high standard deviation naturally also means high downside risk. In particular, risk-averse investors will not like a high standard deviation, i.e., high fluctuation of their returns.

An informative performance measure of an asset or a portfolio is the risk-adjusted return, i.e., the return/risk ratio. For a portfolio it is μ_P/σ_P, which we derived in equations (1.1) and (1.5). In Figure 1.3 we observe one of the few free lunches in finance: the lower (preferably negative) the correlation of the assets in a portfolio, the higher the return/risk ratio. For a rigorous proof, see Markowitz (1952) and Sharpe (1964).

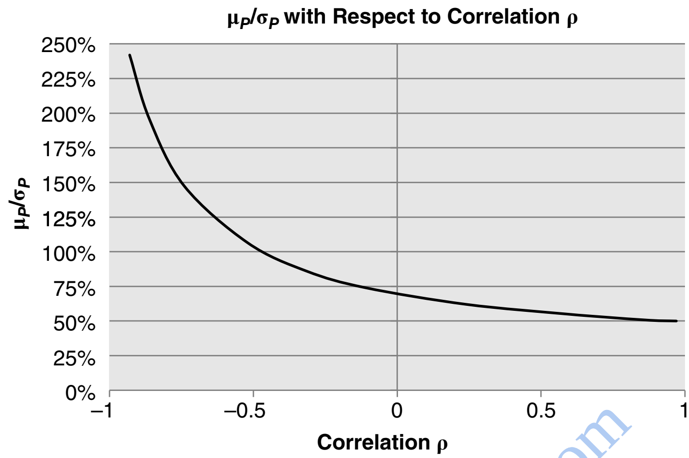

FIGURE 1.3 The Negative Relationship of the Portfolio Return/Risk Ratio μ_P/σ_P with Respect to the Correlation ρ of the Assets in the Portfolio (Input Data are from Table 1.1)

Figure 1.3 shows the high impact of correlation on the portfolio return/risk ratio. A high negative correlation results in a return/risk ratio of close to 250%, whereas a high positive correlation results in a 50% ratio. The equations (1.1) to (1.5) are derived within the framework of the Pearson correlation approach. We will discuss the limitations of this approach in Chapter 3.

> *Only by great risks can great results be achieved.*
>
> —Xeres

1.3.2 Trading and Correlation

In finance, every risk is also an opportunity. Therefore, at every major investment bank and hedge fund *correlation desks* exist. The traders try to forecast changes in correlation and attempt to financially gain from these changes in correlation. We already mentioned the correlation strategy "pairs trading." Generally, *correlation trading* means trading assets whose prices are determined at least in part by the comovement of one or more asset in time. Many types of correlation assets exist.

1.3.2.1 Multi-Asset Options A popular group of correlation options are multi-asset options, also termed rainbow options or mountain range options. Many different types are traded. The most popular ones are listed here. S_1 is the price of asset 1 and S_2 is the price of asset 2 at option maturity. K is the strike price, i.e., the price determined at option start, at which the underlying asset can be bought in the case of a call, and the price at which the underlying asset can be sold in the case of a put.

- Option on the better of two. Payoff = $\max(S_1, S_2)$.
- Option on the worse of two. Payoff = $\min(S_1, S_2)$.
- Call on the maximum of two. Payoff = $\max[0, \max(S_1, S_2) - K]$.
- Exchange option (as a convertible bond). Payoff = $\max(0, S_2 - S_1)$.
- Spread call option. Payoff = $\max[0, (S_2 - S_1) - K]$.
- Option on the better of two or cash. Payoff = $\max(S_1, S_2, \text{cash})$.
- Dual-strike call option. Payoff = $\max(0, S_1 - K_1, S_2 - K_2)$.
- Portfolio of basket options. Payoff = $\left[\sum_{i=1}^{n} n_i S_i - K, 0\right]$, where n_i is the weight of assets i.

Importantly, the prices of these correlation options are highly sensitive to the correlation between the asset prices S_1 and S_2. In the list above, except for the option on the worse of two, the lower the correlation, the higher the option price. This makes sense since a low, preferable negative correlation means that if one asset decreases, on average the other increases. So one of the two assets is likely to result in a high price and a high payoff. Multi-asset options can be conveniently priced using closed form extensions of the Black-Scholes-Merton 1973 option model; see Chapter 9 for details.

Let's look at the evaluation of an exchange option with a payoff of $\max(0, S_2 - S_1)$. The payoff shows that the option buyer has the right to give away asset 1 and receive asset 2 at option maturity. Hence, the option buyer will exercise her right if $S_2 > S_1$. The price of the exchange option can be derived easily. We first rewrite the payoff equation $\max(0, S_2 - S_1) = S_1 \max[0, (S_2/S_1) - 1]$. We then input the covariance between asset S_1 and S_2 into the implied volatility function of the exchange option using a variation of equation (1.5):

$$\sigma_E = \sqrt{\sigma_A^2 + \sigma_B^2 - 2\mathrm{Cov}_{AB}} \tag{1.5a}$$

where σ_E is the implied volatility of S_2/S_1, which is input into the standard Black-Scholes-Merton 1973 option pricing model.

For an exchange option pricing model and further discussion, see Chapter 9, section 9.2.2 and the model "Exchange option.xls" at www.wiley .com/go/correlationriskmodeling, under "Chapter 1."

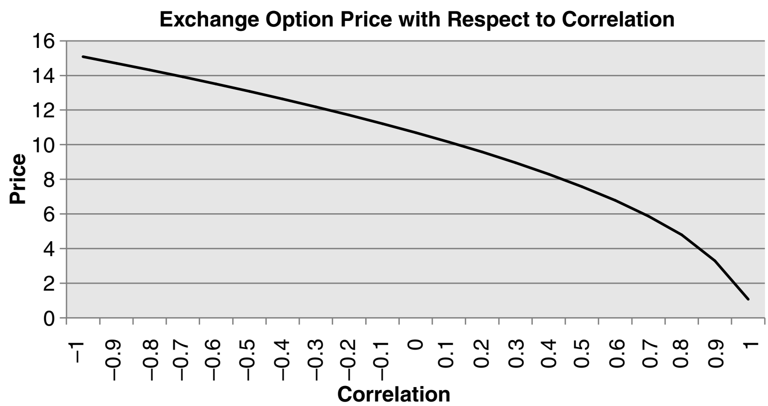

FIGURE 1.4 Exchange Option Price with Respect to Correlation of the Assets in the Portfolio
For details on an exchange option as pricing and correlation risk management, see Chapter 9, section 9.2.2.

Importantly, the exchange option price is highly sensitive to the correlation between the asset prices S_1 and S_2, as seen in Figure 1.4.

From Figure 1.4 we observe the strong impact of the correlation on the exchange option price. The price is close to 0 for high correlation and \$15.08 for a negative correlation of −1. As in Figures 1.2 and 1.3, the correlation approach underlying Figure 1.4 is the Pearson correlation model. We will discuss the limitations of the Pearson correlation model in Chapter 3.

1.3.2.2 Quanto Option Another interesting correlation option is the quanto option. This is an option that allows a domestic investor to exchange his potential option payoff in a foreign currency back into his home currency at a fixed exchange rate. A quanto option therefore protects an investor against currency risk. For example, an American believes the Nikkei will increase, but she is worried about a decreasing yen, which would reduce or eliminate her profits from the Nikkei call option. The investor can buy a quanto call on the Nikkei, with the yen payoff being converted into dollars at a fixed (usually the spot) exchange rate.

Originally, the term *quanto* comes from the word *quantity*, meaning that the amount that is reexchanged to the home currency is unknown, because it depends on the future payoff of the option. Therefore the financial institution that sells a quanto call does not know two things:

1. How deep in the money the call will be, i.e., which yen amount has to be converted into dollars.

2. The exchange rate at option maturity at which the stochastic yen payoff will be converted into dollars.

The correlation between (1) and (2) i.e., the price of the underlying S' and the exchange rate X, significantly influences the quanto call option price. Let's consider a call on the Nikkei S' and an exchange rate X defined as domestic currency per unit of foreign currency (so \$/1 yen for a domestic American) at maturity.

If the correlation is positive, an increasing Nikkei will also mean an increasing yen. That is favorable for the call seller. She has to settle the payoff, but only needs a small yen amount to achieve the dollar payment. Therefore, the more positive the correlation coefficient, the lower the price for the quanto option. If the correlation coefficient is negative, the opposite applies: If the Nikkei increases, the yen decreases in value. Therefore, more yen are needed to meet the dollar payment. As a consequence, the lower the correlation coefficient, the more expensive the quanto option. Hence we have a similar negative relationship between the option price and correlation as in Figure 1.2.

Quanto options can be conveniently priced closed form applying an extension of the Black-Scholes-Merton 1973 model. For a pricing model and a more detailed discussion on a quanto option, see the "Quanto option.xls" model at www.wiley.com/go/correlationriskmodeling under "Chapter 1."

1.3.2.3 Correlation Swap The correlation between assets can also be traded directly with a correlation swap. In a correlation swap a fixed (i.e., known) correlation is exchanged with the correlation that will actually occur, called realized or stochastic (i.e., unknown) correlation, as seen in Figure 1.5.

Paying a fixed rate in a correlation swap is also called *buying correlation*. This is because the present value of the correlation swap will increase for the correlation buyer if the realized correlation increases. Naturally the fixed rate receiver is *selling correlation*.

The realized correlation ρ in Figure 1.5 is the correlation between the assets that actually occurs during the time of the swap. It is calculated as:

$$\rho_{\text{realized}} = \frac{2}{n^2 - n} \sum_{i>j} \rho_{i,j} \tag{1.6}$$

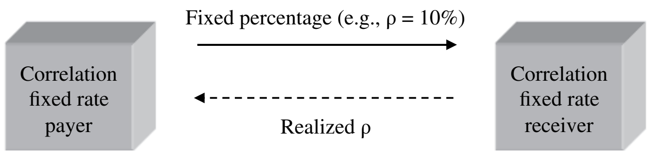

FIGURE 1.5 A Correlation Swap with a Fixed 10% Correlation Rate

where $\rho_{i,j}$ is the Pearson correlation between asset i and j, and n is the number of assets in the portfolio.

The payoff of a correlation swap for the correlation fixed rate payer at maturity is:

$$N\,(\rho_{\text{realized}} - \rho_{\text{fixed}}) \tag{1.7}$$

where N is the notional amount. Let's look at an example of a correlation swap.

Correlation swaps can indirectly protect against decreasing stock prices. As we will see in this chapter in section 1.4, as well as in Chapter 2, when stocks decrease, typically the correlation between the stocks increases. Hence a fixed correlation payer protects himself indirectly against a stock market decline.

EXAMPLE 1.1: PAYOFF OF A CORRELATION SWAP

What is the payoff of a correlation swap with three assets, a fixed rate of 10%, a notional amount of \$1,000,000, and a 1-year maturity?

First, the daily log returns $\ln(S_t/S_{t-1})$ of the three assets are calculated for 1 year.[1] Let's assume the realized pairwise correlations of the log returns at maturity are as displayed in Table 1.2.

The average correlation between the three assets is derived by equation (1.6). We apply the correlations only in the shaded area from Table 1.2, since these satisfy $i > j$. Hence we have $\rho_{\text{realized}} = \frac{2}{3^2-3}(0.5+0.3+0.1) = 0.3$. Following equation (1.7), the payoff for the correlation fixed rate payer at swap maturity is \$1,000,000 × (0.3 − 0.1) = \$200,000.

TABLE 1.2 Pairwise Pearson Correlation Coefficient at Swap Maturity

	$S_{j=1}$	$S_{j=2}$	$S_{j=3}$
$S_{i=1}$	1	0.5	0.1
$S_{i=2}$	0.5	1	0.3
$S_{i=3}$	0.1	0.3	1

1. Log returns $\ln(S_1/S_0)$ are an approximation of percentage returns $(S_1 - S_0)/S_0$. We typically use log returns in finance since they are additive in time, whereas percentage returns are not. For details see Appendix 1B.

Currently, year 2013, there is no industry-standard valuation model for correlation swaps. Traders often use historical data to anticipate $\rho_{realized}$. In order to apply swap valuation techniques, we require a term structure of correlation in time. However, no correlation term structure currently exists. We can also apply stochastic correlation models to value a correlation swap. Stochastic correlation models are currently emerging and will be discussed in Chapter 8.

1.3.2.4 Buying Call Options on an Index and Selling Call Options on Individual Components Another way of buying correlation (i.e., benefiting from an increase in correlation) is to buy call options on an index such as the Dow Jones Industrial Average (the Dow) and sell call options on individual stocks of the Dow. As we will see in Chapter 2, there is a positive relationship between correlation and volatility. Therefore, if correlation between the stocks of the Dow increases, so will the implied volatility[2] of the call on the Dow. This increase is expected to outperform the potential loss from the increase in the short call positions on the individual stocks.

Creating exposure on an index and hedging with exposure on individual components is exactly what the "London whale," JPMorgan's London trader Bruno Iksil, did in 2012. Iksil was called the London whale because of his enormous positions in credit default swaps (CDSs).[3] He had sold CDSs on an index of bonds, the CDX.NA.IG.9, and hedged them by buying CDSs on individual bonds. In a recovering economy this is a promising trade: Volatility and correlation typically decrease in a recovering economy. Therefore, the sold CDSs on the index should outperform (decrease more than) the losses on the CDSs of the individual bonds.

But what can be a good trade in the medium and long term can be disastrous in the short term. The positions of the London whale were so large that hedge funds short-squeezed him: They started to aggressively buy the CDS index CDX.NA.IG.9. This increased the CDS values in the index and created a huge (paper) loss for the whale. JPMorgan was forced to buy back the CDS index positions at a loss of over $2 billion.

2. Implied volatility is volatility derived (implied) by option prices. The higher the implied volatility, the higher the option price.

3. Simply put, a credit default swap (CDS) is an insurance against default of an underlying (e.g., a bond). However, if the underlying is not owned, a long CDS is a speculative instrument on the default of the underlying (just like a naked put on a stock is a speculative position on the stock going down). See Meissner (2005) for more.

1.3.2.5 Paying Fixed in a Variance Swap on an Index and Receiving Fixed on Individual Components A further way to buy correlation is to pay fixed in a variance swap on an index and to receive fixed in variance swaps on individual components of the index. The idea is the same as the idea with respect to buying a call on an index and selling a call on the individual components: If correlation increases, so will the variance. As a consequence, the present value for the variance swap buyer, the fixed variance swap payer, will increase. This increase is expected to outperform the potential losses from the short variance swap positions on the individual components.

In the preceding trading strategies, the correlation between the assets was assessed with the Pearson correlation approach. As mentioned, we will discuss the limitations of this model in Chapter 3.

1.3.3 Risk Management and Correlation

After the global financial crisis from 2007 to 2009, financial markets have become more risk averse. Commercial banks, investment banks, as well as nonfinancial institutions have increased their risk management efforts. As in the investment and trading environment, correlation plays a vital part in risk management. Let's first clarify what risk management means in finance.

Financial risk management is the process of identifying, quantifying, and, if desired, reducing financial risk.

The three main types of financial risk are:

1. Market risk.
2. Credit risk.
3. Operational risk.

Additional types of risk may include systemic risk, liquidity risk, volatility risk, and the topic of this book, correlation risk. We will concentrate in this introductory chapter on market risk. Market risk consists of four types of risk: (1) equity risk, (2) interest rate risk, (3) currency risk, and (4) commodity risk.

There are several concepts to measure the market risk of a portfolio, such as value at risk (VaR), expected shortfall (ES), enterprise risk management (ERM), and more. VaR is currently (year 2013) the most widely applied risk management measure. Let's show the impact of asset correlation on VaR.[4]

First, what is value at risk (VaR)? VaR measures the maximum loss of a portfolio with respect to a certain probability for a certain time frame. The equation for VaR is:

$$\mathrm{VaR}_P = \sigma_P \, \alpha \sqrt{x} \tag{1.8}$$

4. We use a variance-covariance VaR approach in this book to derive VaR. Another way to derive VaR is the nonparametric VaR. This approach derives VaR from simulated historical data. See Markovich (2007) for details.

where VaR_P is the value at risk for portfolio P, and α is the abscise value of a standard normal distribution corresponding to a certain confidence level. It can be derived as =normsinv(confidence level) in Excel or norminv (confidence level) in MATLAB. α takes the values $-\infty < \alpha < +\infty$. x is the time horizon for the VaR, typically measured in days; σ_P is the volatility of the portfolio P, which includes the correlation between the assets in the portfolio. We calculate σ_P via

$$\sigma_P = \sqrt{\beta_h C \beta_v} \qquad (1.9)$$

where β_h is the horizontal β vector of invested amounts (price time quantity), β_v is the vertical β vector of invested amounts (also price time quantity),[5] and C is the covariance matrix of the returns of the assets.

Let's calculate VaR for a two-asset portfolio and then analyze the impact of different correlations between the two assets on VaR.

EXAMPLE 1.2: DERIVING VaR OF A TWO-ASSET PORTFOLIO

What is the 10-day VaR for a two-asset portfolio with a correlation coefficient of 0.7, daily standard deviation of returns of asset 1 of 2%, of asset 2 of 1%, and \$10 million invested in asset 1 and \$5 million invested in asset 2, on a 99% confidence level?

First, we derive the covariances (Cov):

$$\begin{aligned} Cov_{11} &= \rho_{11}\,\sigma_1\,\sigma_1 = 1 \times 0.02 \times 0.02 = 0.0004^6 \\ Cov_{12} &= \rho_{12}\,\sigma_1\,\sigma_2 = 0.7 \times 0.02 \times 0.01 = 0.00014 \\ Cov_{21} &= \rho_{21}\,\sigma_2\,\sigma_1 = 0.7 \times 0.01 \times 0.02 = 0.00014 \\ Cov_{22} &= \rho_{22}\,\sigma_2\,\sigma_2 = 1 \times 0.01 \times 0.01 = 0.0001 \end{aligned} \qquad (1.10)$$

(continued)

5. More mathematically, the vector β_h is the transpose of the vector β_v, and vice versa: ${\beta_h}^T = \beta_v$ and ${\beta_v}^T = \beta_h$. Hence we can also write equation (1.9) as $\sigma_P = \sqrt{\beta_h C {\beta_h}^T}$. See the spreadsheet "Matrix primer.xls," sheet "Matrix Transpose," at www.wiley.com/go/correlationriskmodeling, under "Chapter 1."

6. The attentive reader realizes that we calculated the covariance differently in equation (1.3). In equation (1.3) we derived the covariance from scratch, inputting the return values and means. In equation (1.10) we are assuming that we already know the correlation coefficient ρ and the standard deviation σ.

(*continued*)

Hence our covariance matrix is $C = \begin{pmatrix} 0.0004 & 0.00014 \\ 0.00014 & 0.0001 \end{pmatrix}$

Let's calculate σ_P following equation (1.9). We first derive $\beta_h C$

$$(10\ 5)\begin{pmatrix} 0.0004 & 0.00014 \\ 0.00014 & 0.0001 \end{pmatrix} = (10 \times 0.0004 + 5 \times 0.00014\ 10 \times 0.00014 + 5 \times 0.0001) = (0.0047\ 0.0019)$$

and then $(\beta_h C)\beta_v = (0.0047 \quad 0.0019)\begin{pmatrix} 10 \\ 5 \end{pmatrix} = 10 \times 0.0047 + 5 \times 0.0019 = 5.65\%$[7]

Hence we have $\sigma_P = \sqrt{\beta_h C \beta_v} = \sqrt{5.65\%} = 23.77\%$.

We find the value for α in equation (1.8) from Excel as = normsinv (0.99) = 2.3264, or from MATLAB as norminv(0.99) = 2.3264.

Following equation (1.8), we now calculate the VaR_P as $0.2377 \times 2.3264 \times \sqrt{10} = 1.7486$.

Interpretation: We are 99% certain that we will not lose more than $1.75486 million in the next 10 days due to market price changes of asset 1 and 2.

The number $1.7486 million is the 10-day VaR on a 99% confidence level. This means that on average once in a hundred 10-day periods (so once every 1,000 days) this VaR number of $1.7486 million will be exceeded. If we have roughly 250 trading days in a year, the company is expected to exceed the VaR about once every four years. The Basel Committee for Banking Supervision (BCBS) considers this to be too often. Hence, it requires banks, which are allowed to use their own models (called internal model-based approach), to hold capital for assets in the trading book[8] in the amount of at least 3 times the 10-day VaR (plus a specific risk charge for credit risk). In example 1.2, if a bank is granted the minimum of 3 times the VaR, a VaR

7. The spreadsheet "2-asset VaR.xlsx," which derives the values in example 1.2, can be found at www.wiley.com/go/correlationriskmodeling, section under "Chapter 1."

8. Assets that are marked-to-market, such as stocks, futures, options, and swaps, are in the trading book. Some assets, such as loans and certain bonds, which are not marked-to-market, are in the banking book.

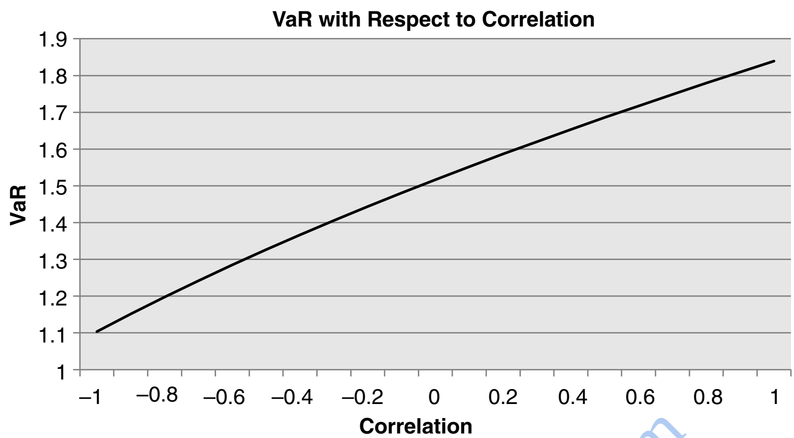

FIGURE 1.6 VaR of the Two-Asset Portfolio of Example 1.2 with Respect to Correlation ρ between Asset 1 and Asset 2

capital charge for assets in the trading book of \$1,7486 million × 3 = \$5.2539 million is required by the Basel Committee.[9]

Let's now analyze the impact of different correlations between the asset 1 and asset 2 on VaR. Figure 1.6 shows the impact.

As expected, we observe from Figure 1.6 that the lower the correlation, the lower the risk, measured by VaR. Preferably the correlation is negative. In this case, if one asset decreases, the other asset on average increases, hence reducing the overall risk. The impact of correlation on VaR is strong. For a perfect negative correlation of −1, VaR is \$1.1 million; for a perfect positive correlation, VaR is close to \$1.9 million.

9. In a recent Consultative Document (May 2012), the Basel Committee has indicated that it is considering replacing VaR with expected shortfall (ES). Expected shortfall measures tail risk (i.e., the size and probability of losses beyond a certain threshold). See www.bis.org/publ/bcbs219.pdf for details. Loosely speaking, VaR answers the question: "What is the maximum loss in good times?" Expected shortfall answers the question: "What is the loss in bad times?"

There are no toxic assets, just toxic people.

1.3.4 The Global Financial Crisis of 2007 to 2009 and Correlation

Currently, in 2013, the global financial crisis of 2007 to 2009 seems almost like a distant memory. The U.S. stock market has recovered from its low in March 2009 of 6,547 points and has more than doubled to over 15,000. World economic growth is at a moderate 2.5%. However, the U.S. unemployment rate is stubbornly high at around 8% and has not decreased to pre-crisis levels of about 5%. Most important, to fight the crisis, countries engaged in huge stimulus packages to revive their faltering economies. As a result, enormous sovereign deficits are plaguing the world economy. The European debt crisis, with Greece, Cyprus, and other European nations virtually in default, is a major global economic threat. The U.S. debt is also far from benign with a debt-to-GDP ratio of over 80%. One of the few nations that is enjoying these enormous debt levels is China, which is happy buying the debt and taking in the proceeds.

A crisis that brought the financial and economic system worldwide to a standstill is naturally not monocausal but has many reasons. Here are the main ones:

- An extremely benign economic and risk environment from 2003 to 2006 with record low credit spreads, low volatility, and low interest rates.
- Increasing risk taking and speculation of traders and investors who tried to benefit in these presumably calm times. This led to a bubble in virtually every market segment, such as the housing market, mortgage market (especially the subprime mortgage market), stock market, and commodity market. In 2007, U.S. investors had borrowed 470% of the U.S. national income to invest and speculate in the real estate, financial, and commodity markets.
- A new class of structured investment products, such as collateralized debt obligations (CDOs), CDO-squareds, constant-proportion debt obligations (CPDOs), constant-proportion portfolio insurance (CPPI), as well as new products like options on credit default swaps (CDSs), credit indexes, and the like.
- The new copula correlation model, which was trusted naively by many investors and which could presumably correlate the $n(n-1)/2$ assets in a structured product. Most CDOs contained 125 assets. Hence there

are $125(125 - 1)/2 = 7{,}750$ asset correlation pairs to be quantified and managed. For details see Chapters 5 and 6.

- A moral hazard of rating agencies, which were paid by the same companies whose assets they rated. As a consequence, many structured products received AAA ratings and gave the illusion of little price and default risk.
- Risk managers and regulators who lowered their standards in light of the greed and profit frenzy. We recommend an excellent (anonymous) paper in the *Economist*: "A Personal View of the Crisis, Confessions of a Risk Manager."

The topic of this book is correlation risk, so let's concentrate on the correlation aspect of the crisis. Around 2003, two years after the Internet bubble burst, the risk appetite of the financial markets increased, and investment banks, hedge funds, and private investors began to speculate and invest in the stock markets, commodity markets, and especially the real estate market.

In particular, residential mortgages became an investment object. The mortgages were packaged in collateralized debt obligations (CDOs; see Chapter 5 for a detailed discussion), and then sold off to investors nationally and internationally. The CDOs typically consist of several tranches; that is, the investor can choose a particular degree of default risk. The equity tranche holder is exposed to the first 3% of mortgage defaults, the mezzanine tranche holder is exposed to the 3% to 7% of defaults, and so on. The new copula correlation model derived by Abe Sklar in 1959 and transferred to finance by David Li in 2000 could presumably manage the default correlations in the CDOs (see Chapters 5 and 6 for details).

The first correlation-related crisis, which was a forerunner of the major one to come in 2007 to 2009, occurred in May 2005. General Motors was downgraded to BB and Ford was downgraded to BB+, so both companies were now in junk status. A downgrade to junk status typically leads to a sharp bond price decline, since many mutual funds and pension funds are not allowed to hold junk bonds.

Importantly, the correlation of the bonds in CDOs that referenced investment grade bonds decreased, since bonds of different credit qualities are typically lower correlated. This led to huge losses of hedge funds, which had put on a strategy where they were short the equity tranche of the CDO and long the mezzanine tranche of the CDO. Figure 1.7 shows the dilemma. Hedge funds had shorted the equity tranche[10] (0% to 3% in Figure 1.7) to

10. Shorting the equity tranche means being short credit protection or selling credit protection, which means receiving the (high) equity tranche contract spread.

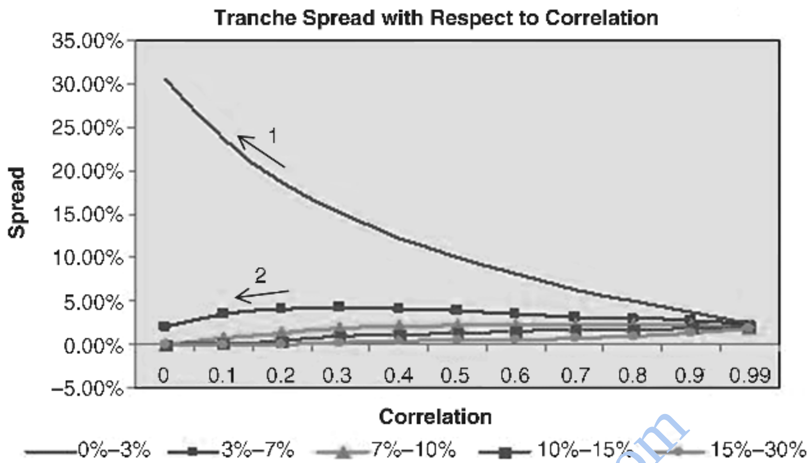

FIGURE 1.7 CDO Tranche Spread with Respect to Correlation

collect the high equity tranche spread. They had then presumably hedged[11] the risk by going long the mezzanine tranche[12] (3% to 7% in Figure 1.7). However, as we can see from Figure 1.7, this hedge is flawed.

When the correlations of the assets in the CDO decreased, the hedge funds lost on both positions.

1. The equity tranche spread increased sharply; see arrow 1. Hence the fixed spread that the hedge funds received in the original transaction was now significantly lower than the current market spread, resulting in a paper loss.
2. In addition, the hedge funds lost on their long mezzanine tranche positions, since a lower correlation lowers the mezzanine tranche spread; see arrow 2. Hence the spread that the hedge funds paid in the original transactions was now higher than the market spread, resulting in another paper loss.

As a result of the huge losses, several hedge funds, such as Marin Capital, Aman Capital, and Baily Coates Cromwell, filed for bankruptcy. It is important to point out that the losses resulted from a lack of understanding of the correlation properties of the tranches in the CDO. The CDOs

11. To hedge means to protect or to reduce risk. See Chapter 11, section 11.1, for details.
12. Going long the mezzanine tranche means being long credit protection or buying credit protection, which means paying the (fairly low) mezzanine tranche contract spread.

themselves can hardly be blamed or be called toxic for their correlation properties.

From 2003 to 2006 the CDO market, mainly referencing residential mortgages, had exploded, increasing from $64 billion to $455 billion. To fuel the CDOs, more and more questionable subprime mortgages were given, named NINJA loans, standing for "no income, no job or assets." When housing prices started leveling off in 2006, the first mortgages started to default. In 2007 more and more mortgages defaulted, finally leading to a real estate market collapse. With it the huge CDO market collapsed, leading to the stock market and commodity market crash and a freeze in the credit markets. The financial crisis spread to the world economies, creating a global severe recession, now called the Great Recession.

In a systemic crash like this, naturally many types of correlations increase (see also Figure 1.8). From 2007 to 2009, default correlations of the mortgages in the CDOs increased. This actually helped equity tranche investors, as we can see from Figure 1.7: If default correlations increase, the equity tranche spread decreases, leading to an increase in the value of the equity tranche. However, this increase was overcompensated by a strong increase in default probability of the mortgages. As a consequence, tranche spreads increased sharply, resulting in huge losses for the equity tranche investors as well as investors in the other tranches.

Correlations between the tranches of the CDOs also increased during the crisis. This had a devastating effect on the super-senior tranches. In normal times, these tranches were considered extremely safe since (1) they were AAA rated and (2) they were protected by the lower tranches. But with the increased tranche correlation and the generally deteriorating credit market, these super-senior tranches were suddenly considered risky and lost up to 20% of their value.

To make things worse, many investors had leveraged the super-senior tranches, termed leveraged super-senior (LSS) tranches, to receive a higher spread. This leverage was typically 10 to 20 times, meaning an investor paid $10,000,000 but had risk exposure of $100,000,000 to $200,000,000. What made things technically even worse was that these LSSs came with an option for the investors to unwind the super-senior tranche if the spread had widened (increased). Many investors started to purchase the LSS spread at very high levels, realizing a loss and increasing the LSS tranche spread even further.

In addition to the overinvestment in CDOs, the credit default swap (CDS) market also exploded from its beginnings in the mid-1990s from about $8 trillion in 2004 to almost $60 trillion in 2007. CDSs are typically used as insurance to protect against default of a debtor, as we explained in Figure 1.1. No one will argue that an insurance contract is toxic. On the contrary, it is the principle of an insurance contract to spread the risk to a wider audience and hence reduce individual risk, as we can see from health insurance or life insurance contracts.

CDSs, though, can also be used as speculative instruments. For example, the CDS seller (i.e., the insurance seller) hopes that the insured event (e.g., default of a company or credit deterioration of the company) will not occur. In this case the CDS seller keeps the CDS spread (i.e., the insurance premium) as income, as American International Group (AIG) tried to do in the crisis. A CDS buyer who does not own the underlying asset is speculating on the credit deterioration of the underlying asset, just like a naked put option holder speculates on the decline of the underlying asset.

So who is to blame for the 2007–2009 global financial crisis? The quants, who created the new products such as CDSs and CDOs and the models to value them? The upper management and the traders, who authorized and conducted the overinvesting and extreme risk taking? The rating agencies, who gave an AAA rating to many of the CDOs? The regulators, who approved the overinvestments? The risk managers, who allowed the excessive risk taking?

The entire global financial crisis can be summed up in one word: Greed! It was the upper management, the traders, and the investors who engaged in excessive trading and irresponsible risk taking to receive high returns, huge salaries, and generous bonuses. For example, the London unit of AIG had sold close to $500 billion in CDSs without much reinsurance! Their main hedging strategy seemed to have been: Pray that the insured contracts don't deteriorate. The investment banks of the small Northern European country of Iceland had borrowed 10 times Iceland's national GDP and invested it. With this leverage, Iceland naturally went de facto into bankruptcy in 2008, when the credit markets deteriorated. Lehman Brothers, before filing for bankruptcy in September 2008, reported a leverage of 30.7 (i.e., $691 billion in assets and only $22 billion in stockholders' equity). The true leverage was even higher, since Lehman tried to hide the leverage with materially misleading repo transactions.[13] In addition, Lehman had 1.5 million derivatives transactions with 8,000 different counterparties on its books.

Did the upper management and traders of hedge funds and investment banks admit to their irresponsible leverage, excessive trading, and risk taking? No. Instead they created the myth of the toxic asset, which is absurd.

13. *Repo* stands for repurchase transaction. It can be viewed as a short-term collateralized loan.

It is like a murderer saying, "I did not shoot that person. It was my gun!" Toxic are not the financial products, but humans and their greed.

Most traders were well aware of the risks that they were taking. In the few cases where traders did not understand the risks, the asset itself cannot be blamed; rather, the incompetence of the trader is the reason for the loss. While it is ethically disappointing that the investors and traders did not admit to their wrongdoing, at the same time it is understandable. If they admitted to irresponsible trading and risk taking, they would immediately be prosecuted.

Naturally, risk managers and regulators have to take part of the blame for allowing the irresponsible risk taking. The moral hazard of the rating agencies, being paid by the same companies whose assets they rate, also needs to be addressed.

We will discuss the role of financial models, their benefits, and their limitations at the beginning of Chapter 3.

1.3.5 Regulation and Correlation

Correlations are critical inputs in regulatory frameworks such as the Basel accords, especially in regulations for market risk and credit risk. We will discuss the correlation approaches of the Basel accords in this book. First, let's clarify.

1.3.5.1 What Are Basel I, II, and III? Basel I, implemented in 1988; Basel II, implemented in 2006; and Basel III, which is currently being developed and implemented until 2018, are regulatory guidelines to ensure the stability of the banking system.

The term *Basel* comes from the beautiful city of Basel in Switzerland, where the honorable regulators meet. None of the Basel accords has legal authority. However, most countries (about 100 for Basel II) have created legislation to enforce the Basel accords for their banks.

1.3.5.2 Why Basel I, II, and III? The objective of the Basel accords is to "provide incentives for banks to enhance their risk measurement and management systems" and "to contribute to a higher level of safety and soundness in the banking system." In particular, Basel III is being developed to address the deficiencies of the banking system during the financial crisis of 2007 to 2009. Basel III introduces many new ratios to ensure liquidity and adequate leverage of banks. In addition, new correlation models will be implemented that deal with double defaults in insured risk transactions as displayed in Figure 1.1. Correlated defaults in a multi-asset portfolio quantified with the Gaussian copula, correlations in derivatives transactions termed credit value adjustment (CVA), and correlations in what is called wrong-way risk (WWR) are currently being discussed. We devote the entire Chapter 12 to addressing the benefits and limitations of these correlation approaches in Basel III.

1.4 HOW DOES CORRELATION RISK FIT INTO THE BROADER PICTURE OF RISKS IN FINANCE?

As already mentioned in section 1.3.3, we differentiate three main types of risks in finance:

1. Market risk
2. Credit risk
3. Operational risk

Additional types of risk may include systemic risk, concentration risk, liquidity risk, volatility risk, legal risk, reputational risk, and more. Correlation risk plays an important part in market risk and credit risk, and is closely related to systemic risk and concentration risk. Let's discuss it.

1.4.1 Correlation Risk and Market Risk

Correlation risk is an integral part of market risk. Market risk, comprised of equity risk, interest rate risk, currency risk, and commodity risk. Market risk is typically measured with the value at risk (VaR) concept. VaR has a covariance matrix of the assets in the portfolio as an input. So market risk implicitly incorporates correlation risk, i.e., the risk that the correlations in the covariance matrix change. We have already studied the impact of different correlations on VaR in section 1.3.3, "Risk Management and Correlation."

Market risk is also quantified with expected shortfall (ES), also termed conditional VaR or tail risk. Expected shortfall measures market risk for extreme events, typically for the worst 0.1%, 1%, or 5% of possible future scenarios. A rigorous valuation of expected shortfall naturally includes the correlation between the asset returns in the portfolio, as VaR does.[14]

14. Unfortunately, different authors use different definitions (and notation) for ES. To study ES, we recommend the original ES paper by Artzner et al. (1997), an educational paper by Yamai and Yoshiba (2002), as well as Acerbi and Tasche (2001) and McNeil, Frey, and Embrechts (2005).

1.4.2 Correlation Risk and Credit Risk

Correlation risk is also a critical part of credit risk. Credit risk is comprised of (1) migration risk and (2) default risk. Migration risk is the risk that the credit quality of a debtor decreases, i.e., migrates to a lower credit state. A lower credit state typically results in a lower asset price, so a paper loss for the creditor. We already studied in section 1.2 the effect of correlation risk of an investor who has hedged his bond exposure with a CDS. We derived that the investor is exposed to the correlation between the reference asset and the counterparty, the CDS seller. The higher the correlation, the higher the CDS paper loss for the investor and, importantly, the higher the probability of a total loss of the investment.

The degree to which defaults occur together (i.e., default correlation) is critical for financial lenders such as commercial banks, credit unions, mortgage lenders, and trusts, which give many types of loans to companies and individuals. Default correlations are also critical for insurance companies, which are exposed to credit risk of numerous debtors. Naturally, a low default correlation of debtors is desired to diversify the credit risk. Table 1.3 shows the default correlation from 1981 to 2001 of 6,907 companies, of which 674 defaulted.

The default correlations in Table 1.3 are one-year default correlations averaged over the time period 1981 to 2001. We will see how to calculate default correlations in Chapter 4, especially in section 4.2, "The Binomial Correlation Measure" (Lucas 1995).

From Table 1.3, we observe that default correlations between industries are mostly positive with the exception of the energy sector. This sector is typically viewed as a recession-resistant, stable sector with little or no correlation to other sectors. We also observe that the default correlation within sectors is higher than between sectors. This suggests that systematic factors (e.g., a recession or structural weakness such as the general decline of a sector) have a greater impact on defaults than do idiosyncratic factors. Hence if General Motors defaults, it is more likely that Ford will default, rather than Ford benefiting from the default of its rival.

Since the intrasector default correlations are higher than intersector default correlations, a lender is advised to have a sector-diversified loan portfolio to reduce default correlation risk.

Defaults are binomial events, either default or no default. So principally we can use a simple correlation model such as the binomial model of Lucas (1995) to analyze them, which we will do in Chapter 4, section 4.2. However, we can also analyze defaults in more detail and look at term structure of defaults. Let's assume a creditor has given loans to two debtors. One debtor is

TABLE 1.3 Default Correlation of 674 Defaulted Companies by Industry

One-Year U.S. Default Correlations—Non-Investment-Grade Bonds 1981–2001

	Auto	Cons	Ener	Fin	Build	Chem	HiTech	Insur	Leis	Tele	Trans	Util
Auto	3.80%	1.30%	1.20%	0.40%	1.10%	1.60%	2.80%	−0.50%	1.00%	3.90%	1.30%	0.50%
Cons	1.30%	2.80%	−1.40%	1.20%	2.80%	1.60%	1.80%	1.10%	1.30%	3.20%	1.30%	1.90%
Ener	1.20%	−1.40%	6.40%	−2.50%	−0.50%	0.40%	−0.10%	−1.60%	−1.00%	−1.40%	−0.10%	0.70%
Fin	0.40%	1.20%	−2.50%	**5.20%**	2.60%	0.10%	2.30%	3.00%	1.60%	3.70%	1.50%	4.50%
Build	1.10%	2.80%	−0.50%	2.60%	**6.10%**	1.20%	2.30%	1.80%	2.30%	**6.50%**	4.20%	1.30%
Chem	1.60%	1.60%	0.40%	0.10%	1.20%	3.20%	1.40%	−1.10%	1.10%	2.80%	1.10%	1.00%
HiTech	2.80%	1.80%	−0.10%	0.40%	2.30%	1.40%	3.30%	0.00%	1.10%	2.80%	1.10%	1.00%
Insur	−0.50%	1.10%	−1.60%	3.00%	1.80%	−1.10%	0.00%	**5.60%**	1.20%	−2.60%	2.30%	1.40%
Leis	1.00%	1.30%	−1.00%	1.60%	2.30%	1.10%	1.40%	1.20%	2.30%	4.00%	2.30%	0.60%
Tele	3.90%	3.20%	−1.40%	3.70%	**6.50%**	2.80%	4.70%	−2.60%	4.00%	**10.70%**	3.20%	−0.80%
Trans	1.30%	2.70%	−0.10%	1.50%	4.20%	1.10%	1.90%	2.30%	2.30%	3.20%	4.30%	−0.20%
Util	0.50%	1.90%	0.70%	4.50%	1.30%	1.00%	1.00%	1.40%	0.60%	−0.80%	−0.20%	**9.40%**

Correlations above 5% are bold.

Source: Standard & Poor's (S&P) 500.

TABLE 1.4 Term Structure of Default Probabilities for an A-Rated Bond and a CC-Rated Bond in 2002

	Year									
	1	2	3	4	5	6	7	8	9	10
A	0.02%	0.07%	0.13%	0.14%	0.15%	0.17%	0.18%	0.21%	0.24%	0.25%
CC	23.83%	13.29%	10.31%	7.62%	5.04%	5.13%	4.04%	4.62%	2.62%	2.04%

Source: Moody's.

A rated, and one is CC rated. A historical default term structure these bonds is displayed in Table 1.4.

For most investment grade bonds, the term structure of default probabilities increases in time, as we see from Table 1.4 for the A-rated bond. This is because the longer the time horizon, the higher the probability of adverse internal events such as mismanagement, or adverse external events such as increased competition or a recession. For bonds in distress, however, the default term structure is typically inverse, as seen for the CC-rated bond in Table 1.4. This is because for a distressed company the immediate future is critical. If the company survives the coming problematic years, the probability of default decreases.

For a creditor, the default correlation of her debtors is critical. As mentioned, a creditor will benefit from a low default correlation of her debtors, which spreads the default correlation risk. We can correlate the default term structures in Table 1.4 with the famous (now infamous) copula model, which will be discussed in Chapter 4. This will allow us to answer such questions as: "What is the joint probability of debtor 1 defaulting in year 3 and debtor 2 defaulting in year 5?"

> *Correlations always increase in stressed markets.*
>
> —John Hull

1.4.3 Correlation Risk and Systemic Risk

So far, we have analyzed correlation risk with respect to market risk and credit risk and have concluded that correlations are a critical input when quantifying market risk and credit risk. Correlations are also closely related to systemic risk, which we define here.

SYSTEMIC RISK

The risk of a financial market or an entire financial system collapsing.

An example of systemic risk is the collapse of the entire credit market in 2008. At the height of the crisis in September 2008, when Lehman Brothers filed for bankruptcy, the credit markets were virtually frozen with essentially no lending activities. Even as the Federal Reserve guaranteed interbank loans, lending resumed only very gradually and slowly.

The stock market crash starting in October 2007 with the Dow Jones Industrial Average at 14,093 points and then falling by 53.54% to 6,547 points by March 2009 is also a systemic market collapse. All but one of the Dow 30 stocks had declined. Walmart was the lone Dow stock that was up during the crisis. Of the S&P 500 stocks, 489 declined during this time frame. The 11 stocks that were up were:

1. Apollo Group (APOL), educational sector; provides educational programs for working adults and is a subsidiary of the University of Phoenix.
2. AutoZone (AZO), auto industry; provides auto replacement parts.
3. CF Industries (CF), agricultural industry; provides fertilizer.
4. DeVry Inc. (DV), educational sector; holding company of several universities.
5. Edward Lifesciences (EW), pharmaceutical industry; provides products to treat cardiovascular diseases.
6. Family Dollar (FDO), consumer staples.
7. Gilead Pharmaceuticals (GILD), pharmaceutical industry; provides HIV, hepatitis medications.
8. Netflix (NFLX), entertainment industry; provides Internet subscription service.
9. Ross Stores (ROST), consumer staples.
10. Southwestern Energy (SWN), energy sector.
11. Walmart (WMT), consumer staples.

From this list we can see that the consumer staples sector (which provides such basic necessities as food and household items) fared well during the crisis. The educational sector also typically thrives in a crisis, since many unemployed seek to further their education.

Importantly, systemic financial failures such as the one from 2007 to 2009 typically spread to the economy, with a decreasing GDP, increasing unemployment, and therefore a decrease in the standard of living.

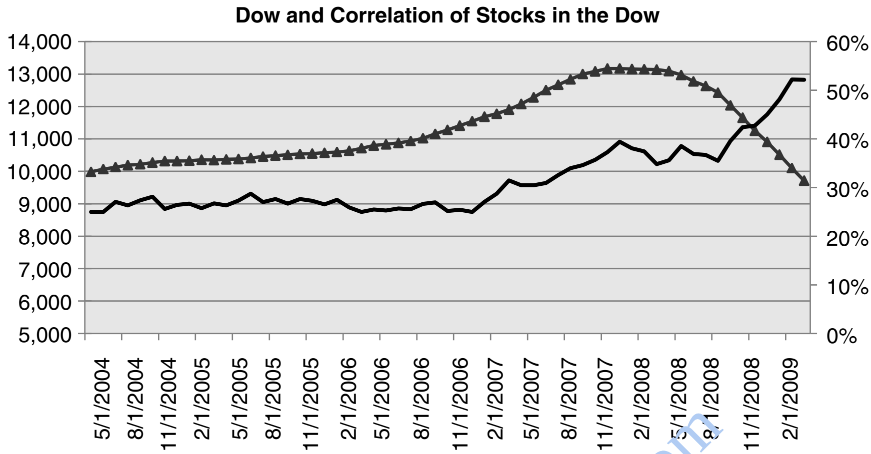

FIGURE 1.8 Relationship between the Dow (graph with triangles, numerical values on left axis) and Correlation between the Stocks in the Dow (numerical values on right axis)

Systemic risk and correlation risk are highly dependent. Since a systemic decline in stocks involves almost the entire stock market, correlations between the stocks increase sharply. Figure 1.8 shows the relationship between the percentage change of the Dow Jones Industrial Average, short "Dow," and the correlation between the stocks in the Dow before the crisis from May 2004 to October 2007 and during the crisis from October 2007 to March 2009.

In Figure 1.8 we downloaded daily closing prices of all 30 stocks in the Dow and put them into monthly bins. We then derived monthly 30 × 30 correlation matrices using the Pearson correlation measure and averaged the matrices. We then smoothed the graph by taking the one-year moving average.

From Figure 1.8 we can observe a somewhat stable correlation from 2004 to 2006, when the Dow increased moderately. In the time period from January 2007 to February 2008 we observe that the correlation in the Dow increases when the Dow increases more strongly. Importantly, in the time of the severe decline of the Dow from August 2008 to March 2009, we observe a sharp increase in the correlation from noncrisis levels of on average 27% to over 50%. In Chapter 2, we will observe empirical correlations in detail, and we will find that at the height of the crisis in February 2009 the correlation of the stocks in the Dow reached a high of 96.97%. Hence, portfolios that were considered well diversified in benign times experienced a sharp increase in correlation and hence unexpected losses due to the combined, highly correlated decline of many stocks during the crisis. We will quantify this correlation risk and its associated potential losses in detail in Chapters 9 and 10.

1.4.4 Correlation Risk and Concentration Risk

Concentration risk is a fairly new risk category and therefore not yet uniquely defined. We provide a sensible definition.

CONCENTRATION RISK

The risk of financial loss due to a concentrated exposure to a particular group of counterparties.

Concentration risk can be quantified with the concentration ratio. For example, if a creditor has 10 loans of equal size, the concentration ratio would be $1/10 = 0.1$. If a creditor has only one loan to one counterparty, the concentration ratio would be 1. Naturally, the lower the concentration ratio, the more diversified is the default risk of the creditor, assuming the default correlation between the counterparties is smaller than 1.

We can also categorize counterparties into groups, for example sectors. We can then analyze sector concentration risk. The higher the number of different sectors a creditor has lent to, the higher is the sector diversification. High sector diversification reduces default risk, since intrasector defaults are more highly correlated than counterparties in different sectors, as seen in Table 1.3.

Naturally, concentration risk and correlation risk are closely related. Let's verify this in an example.

EXAMPLE 1.3: CONCENTRATION RATIO AND CORRELATION

CASE A

The commercial bank *C* has lent \$10,000,000 to a single company, *W*. So *C*'s concentration ratio is 1. Let's assume company *W* has a default probability (P_W) of 10%. Hence the expected loss (EL) for bank *C* is $\$10,000,000 \times 0.1 = \$1,000,000$ (see Figure 1.9).

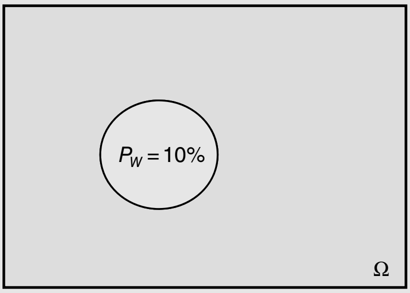

FIGURE 1.9 Probability Space for the Default Probability of a Single Loan to *W*

CASE B

The commercial bank *C* has lent \$5,000,000 to company *X* and \$5,000,000 to company *Y*. Let's assume both *X* and *Y* have a 10% default probability. So *C*'s concentration ratio is reduced to ½.

If the default correlation between *X* and *Y* is bigger than 0 and smaller than 1, we derive that the worst-case scenario [i.e., the default of *X* and *Y*, $P(X \cap Y)$, with a loss of \$1,000,000] is reduced, as seen in Figure 1.10.

The exact joint default probability $P(X \cap Y)$ depends on the correlation model and correlation parameter values, which will be discussed in Chapters 3 to 8. For any model, though, if default correlation between *X* and *Y* is 1, then there is no benefit from the lower concentration ratio. The probability space would have the form as in Figure 1.9.

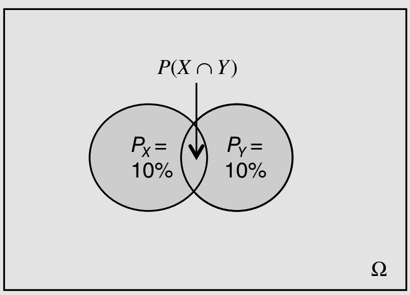

FIGURE 1.10 Probability Space for Loans to Companies *X* and *Y*

(continued)

(*continued*)

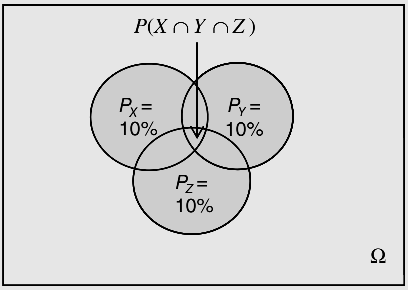

FIGURE 1.11 Probability Space for Loans to Companies X, Y, and Z

CASE C

If we further decrease the concentration ratio, the worst-case scenario (i.e., the expected loss of 10%) decreases further. Let's assume the lender *C* gives loans to three companies, *X*, *Y*, and *Z*, of $3.33 million each. Let's assume that the default probabilities of *X*, *Y*, and *Z* are 10% each. Therefore the concentration ratio decreases to ⅓. The probabilities are displayed in Figure 1.11.

Hence from Figures 1.9 to 1.11 we observe the benefits of a lower concentration ratio. The worst-case scenario, an expected loss of $1,000,000, reduces with a decreasing concentration ratio.

A decreasing concentration ratio is closely related to a decreasing correlation coefficient. Let's show this. The defaults of companies *X* and *Y* are expressed as two binomial variables, which take the value 1 if in default, and 0 otherwise. Equation (1.11) gives the joint probability of default for the two binomial events:

$$P(X \cap Y) = \rho_{XY}\sqrt{P_X(1-P_X)P_Y(1-P_Y)} + P_X P_Y \qquad (1.11)$$

where ρ_{XY} is the correlation coefficient and

$$\sqrt{P_X(1-P_X)} \qquad (1.12)$$

is the standard deviation of the binomially distributed variable *X*.

Let's assume again that the lender *C* has given loans to *X* and *Y* of $5,000,000 each. Both *X* and *Y* have a default probability of 10%. Following equation (1.12), this means that the standard deviation for *X* and *Y* is $\sqrt{0.1 \times (1-0.1)} = 0.3$.

Let's first look at the case where the default correlation is $\rho_{XY} = 1$. This means that X and Y cannot default individually. They can only default together or survive together. The probability that they default together is 10%. Hence the expected loss is the same as in case A: EL = (\$5,000,000 + \$5,000,000) × 0.1 = \$1,000,000. We can verify this with equation (1.11) for the joint probability of two binomial events, $P(X \cap Y) = 1 \times \sqrt{0.1\,(1-0.1) \times 0.1\,(1-0.1)} + 0.1 \times 0.1 = 10\%$. The probability space is graphically the same as Figure 1.9 with $P_X = P_Y = 10\%$ as the probability event.

If we now decrease the correlation coefficient, we can see from equation (1.11) that the worst-case scenario, the joint default probability of X and Y, $P(X \cap Y)$, will decrease. For example, $\rho_{XY} = 0.5$ results in $P(X \cap Y) = 5.5\%$, and $\rho_{XY} = 0$ results in $P(X \cap Y) = 1\%$. Interestingly, even a slightly negative correlation coefficient can result in a positive joint default probability if the standard deviation of the binomial events is fairly low and the default probabilities are high. In our example, the standard deviation of both entities is 30% and a default probability of both entities is 10%. Together with a negative correlation coefficient of −0.1, following equation (1.11), this leads to a joint default probability of 0.1%. We will discuss the binomial correlation model in more detail in Chapter 4, section 4.2.

In conclusion, we have shown the beneficial aspect of a lower concentration ratio, which is closely related to a lower correlation coefficient. In particular, both a lower concentration ratio and a lower correlation coefficient reduce the worst-case scenario for a creditor, the joint probability of default of his debtors.

In Chapter 12, section 12.2, we will verify this result and find that a higher (copula) correlation between assets results in a higher credit value at risk (CVaR). CVaR measures the maximum loss of a portfolio of correlated debt with a certain probability for a certain time frame. Hence CVaR measures correlated default risk and is analogous to the VaR concept for correlated market risk, which we discussed in section 1.3.

1.5 A WORD ON TERMINOLOGY

As mentioned in section 1.3.2, "Trading and Correlation," we find the terms *correlation desks* or *correlation trading* in trading practice. Correlation trading means that traders trade assets or execute trading strategies whose value is at least in part determined by the comovement of two or more assets in time. We already mentioned pairs trading, the exchange option, and the quanto option as examples of correlation trading. In trading practice, the term *correlation* is typically applied quite broadly, referring to any comovement of asset prices in time.

However, in financial theory, especially in recent publications, the term *correlation* is often defined more narrowly, referring only to the linear Pearson correlation model, as in Cherubini, Luciano, and Vecchiato (2004), Nelsen (2006), or Gregory (2010). These authors refer to other than Pearson correlation coefficients as dependence measures or measures of association. However, in financial theory the term *correlation* is also often applied generally to describe dependencies, as in the terms *credit correlation*, *default correlation*, or *volatility-asset return correlation*, which are quantified by non-Pearson models as in Heston (1993), Lucas (1995), or Li (2000).

In this book, we will refer to the Pearson coefficient, discussed in Chapter 3, section 3.1, as correlation coefficient and the coefficients derived by non-Pearson models as dependency coefficients. In accordance with most literature, we will refer to all methodologies that measure some form of dependency as correlation models or dependency models. Ordinal dependence measures, discussed in sections 3.2 and 3.3, which are related to the Pearson correlation approach, will be termed rank correlation measures.

1.6 SUMMARY

There are two types of financial correlations: (1) *Static* correlations measure how two or more financial assets are associated within a certain time period, for example a year. (2) *Dynamic* financial correlations measure how two or more financial assets move together in time.

Correlation risk can be defined as the risk of financial loss due to adverse movements in correlation between two or more variables. These variables can be financial variables such as correlated defaults of two debtors or nonfinancial such as the correlation between political tensions and an exchange rate. Correlation risk can be nonmonotonous, meaning that the dependent variable, for example the CDS spread, can sometimes increase and sometimes decrease when the correlation parameter value increases.

Correlations and correlation risk are critical in many areas in finance, such as investments, trading, and especially in risk management, where different correlations result in very different degrees of risk. Correlations also play a key role in a systemic crisis, where correlations typically increase and can lead to high unexpected losses. As a result, the Basel III accord has introduced several correlation concepts and measures to reduce correlation risk (see Chapter 12 for details).

Correlation risk can be categorized as its own type of risk. However, correlation parameters and correlation matrices are critical inputs and hence a part of market risk and credit risk. Market risk and credit risk are highly sensitive to changing correlations. Correlation risk is also closely related to concentration risk, as well as systemic risk, since correlations typically increase in a systemic crisis.

The term *correlation* is not uniquely defined. In trading practice, *correlation* is applied quite broadly and refers to the comovements of assets in time, which may be measured by different correlation concepts. In financial theory, the term *correlation* is often defined more narrowly, referring only to the linear Pearson correlation coefficient. Non-Pearson correlation measures are termed *dependence measures* or *measures of association*.

APPENDIX 1A: DEPENDENCE AND CORRELATION

Dependence

In statistics, two events are considered dependent if the occurrence of one affects the probability of another. Conversely, two events are considered independent if the occurrence of one does not affect the probability of another. Formally, two events, A and B, are independent if and only if the joint probability equals the product of the individual probabilities:

$$P(A \cap B) = P(A)P(B) \tag{1A.1}$$

Solving equation (1A.1) for $P(A)$, we get

$$P(A) = \frac{P(A \cap B)}{P(B)}$$

Following the Kolmogorov definition $\frac{P(A \cap B)}{P(B)} \equiv P(A|B)$, we derive

$$P(A) = \frac{P(A \cap B)}{P(B)} = P(A|B) \tag{1A.2}$$

where $P(A|B)$ is the conditional probability of A with respect to B. $P(A|B)$ reads “probability of A given B.” In equation (1A.2), the probability of A, $P(A)$, is not affected by B, since $P(A) = P(A|B)$; hence the event A is independent from B.

From equation (1A.2) we also derive

$$P(B) = \frac{P(A \cap B)}{P(A)} = P(B|A) \tag{1A.3}$$

Hence from equation (1A.1) it follows that A is independent from B and B is independent from A.

Example of Statistical Independence The historical default probability of company A is $P(A) = 3\%$, the historical default probability of company B is $P(B) = 4\%$, and the historical joint probability of default is $3\% \times 4\% = 0.12\%$. In this case $P(A)$ and $P(B)$ are independent. This is because from equation (1A.2), we have

$$P(A) = \frac{P(A \cap B)}{P(B)} = P(A|B) = 3\% = \frac{3\% \times 4\%}{4\%} = 3\%$$

Since $P(A) = P(A|B)$, the default probability of company A is independent from the default probability of company B. Using equation (1A.3), we can do the same exercise for the default probability of company B, which is independent from the default probability of company A.

Correlation

As mentioned in section 1.5, the term *correlation* is not uniquely defined. In trading practice, the term *correlation* is used quite broadly, referring to any comovement of asset prices in time. In statistics, correlation is typically defined more narrowly and typically refers to the linear dependency derived in the Pearson correlation model. Let's look at the Pearson covariance and relate it to the dependence discussed earlier.

A covariance measures how strong the linear relationship between two variables is. These variables can be deterministic (meaning their outcome is known), as the historical default probabilities in equation 1A.1. For random variables (variables with an unknown outcome such as flipping a coin), the Pearson covariance is derived with expectation values:

$$\mathrm{Cov}(X, Y) = E[(X - E(X))(Y - E(Y))] = E(XY) - E(X)E(Y) \tag{1A.4}$$

where $E(X)$ and $E(Y)$ are the expected values of (X) and (Y) respectively, also known as the mean, and $E(XY)$ is the expected value of the product of the random variables X and Y.

The covariance in equation (1A.4) is not easy to interpret. Therefore, often a normalized covariance, the correlation coefficient, is used. The Pearson correlation coefficient $\rho(XY)$ is defined as

$$\rho(X, Y) = \frac{\mathrm{Cov}(X, Y)}{\sigma(X)\sigma(Y)} \tag{1A.5}$$

where $\sigma(X)$ and $\sigma(Y)$ are the standard deviations of X and Y, respectively. While the covariance takes values between $-\infty$ and $+\infty$, the correlation coefficient conveniently takes values between -1 and $+1$.

Independence and Uncorrelatedness

From equation (1A.1) above we find that the condition for independence of two random variables is $E(XY) = E(X)\ E(Y)$. From equation (1A.4) we see that $E(XY) = E(X)\ E(Y)$ is equal to a covariance of zero. Therefore, if two variables are independent, their covariance is zero.

Is the reverse also true? Does a zero covariance mean independence? The answer is no. Two variables can have a zero covariance even when they are dependent! Let's show this with an example. For the parabola $Y = X^2$, Y is clearly dependent on X, since Y changes when X changes. However, the correlation of the function $Y = X^2$ derived by equations (1A.4) or (1A.5) is zero! This can be shown numerically and algebraically. For a numerical derivation, see the simple spreadsheet "Dependence and Correlation.xlsm," sheet 1, at www.wiley.com/go/correlationriskmodeling, under "Chapter 1."

Algebraically, we have from equation (1A.4):

$$\mathrm{Cov}(X, Y) = E(XY) - E(X)E(Y)$$

Inputting $Y = X^2$, we derive

$$\begin{aligned} \mathrm{Cov}(X, Y) &= E(X\,X^2) - E(X)E(X^2) \\ &= E(X^3) - E(X)E(X^2) \end{aligned}$$

Let X be a uniform variable bounded in $[-1, +1]$. Then the mean $E(X)$ is zero and we have

$$\begin{aligned} \mathrm{Cov}(X, Y) &= 0 - 0\,E(X^2) \\ &= 0 \end{aligned}$$

For a numerical example, see the spreadsheet "Dependence and Correlation.xlsm," sheet 2, at www.wiley.com/go/correlationriskmodeling under "Chapter 1."

In conclusion, the Pearson covariance or correlation coefficient can give values of zero; that is, it tells us that the variables are uncorrelated, even if the variables are dependent! This is because the Pearson correlation concept measures only linear dependence. It fails to capture nonlinear relationships. This shows the limitation of the Pearson correlation concept for finance, since most financial relationships are nonlinear. See Chapter 3 for a more detailed discussion on the Pearson correlation model.

APPENDIX 1B: ON PERCENTAGE AND LOGARITHMIC CHANGES

In finance, growth rates are expressed as relative changes, $(S_t - S_{t-1})/S_{t-1}$, where S_t and S_{t-1} are the prices of an asset at time t and $t - 1$, respectively. For example, if $S_t = 110$, and $S_{t-1} = 100$, the relative change is $(110 - 100)/100 = 0.1 = 10\%$.

We often approximate relative changes with the help of the natural logarithm:

$$(S_t - S_{t-1})/S_{t-1} \approx \ln(S_t/S_{t-1}) \tag{1B.1}$$

This is a good approximation for small differences between S_t and S_{t-1}. $\mathrm{Ln}(S_t/S_{t-1})$ is called a log return. The advantage of using log returns is that they can be added over time. Relative changes are not additive over time. Let's show this in two examples.

Example 1: A stock price at t_0 is \$100. From t_0 to t_1, the stock increases by 10%. Hence the stock increases to \$110. From t_1 to t_2 the stock increases again by 10%. So the stock price increases to $\$110 \times 0.1 = \$121$. This increase of 21% is higher than adding the percentage increases of $10\% + 10\% = 20\%$. Hence percentage changes are not additive over time.

Let's look at the log returns. The log return from t_0 to t_1 is $\ln(110/100) = 9.531\%$. From t_1 to t_2 the log return is $\ln(121/110) = 9.531\%$. When adding these returns, we get $9.531\% + 9.531\% = 19.062\%$. This is the same as the log return from t_0 to t_2; that is, $\ln(121/100) = 19.062\%$. Hence log returns are additive in time.[15]

Let's now look at another, more extreme example.

Example 2: A stock price in t_0 is \$100. It moves to \$200 in t_1 and back to \$100 in t_2. The percentage change from t_0 to t_1 is $(\$200 - \$100)/\$100 = 100\%$.

15. We could also have solved for the absolute value 121, which matches a logarithmic growth rate of 9.531%: $\ln(x/110) = 9.531\%$, or $\ln(x) - \ln(110) = 9.531\%$, or $\ln(x) = \ln(110) + 9.531\%$. Taking the power of e, we get $e^{(\ln(x))} = x = e^{(\ln(110)+0.09531)} = 121$.

The percentage change from t_1 to t_2 is (\$100 − \$200)/(200) = −50%. Adding the percentage changes, we derive +100% − 50% = +50%, although the stock has not increased from t_0 to t_2! Naturally, this type of performance measure is incorrect and not allowed in accounting.

Log returns give the correct answer: The log return from t_0 to t_1 is ln(200/100) = 69.31%. The log return from t_1 to t_2 is ln(100/200) = −69.31%. Adding these log returns in time, we get the correct return of the stock price from t_0 to t_2 of 69.31% − 69.31% = 0%.

These examples are displayed in the simple spreadsheet "Log returns.xlsx" at www.wiley.com/go/correlationriskmodeling, under "Chapter 1."

PRACTICE QUESTIONS AND PROBLEMS

1. What two types of financial correlations exist?
2. What is wrong-way correlation risk or short wrong-way risk?
3. Correlations can be nonmonotonous. What does this mean?
4. Correlations are critical in many areas in finance. Name five.
5. High diversification is related to low correlation. Why is this considered one of the few free lunches in finance?
6. Create a numerical example and show why a lower correlation results in a higher return/risk ratio.
7. What is correlation trading?
8. What is pairs trading?
9. Name three correlation options in which a lower correlation results in a higher option price.
10. Name one correlation option where a lower correlation results in a higher option price.
11. Create a numerical example of a two-asset portfolio and show that a lower correlation coefficient leads to a lower VaR number.
12. Why do correlations typically increase in a systemic market crash?
13. In 2005, a correlation crisis with respect to CDOs occurred that led to huge losses for several hedge funds. What happened?
14. In the global financial crisis of 2007 to 2009, many investors in the presumably safe super-senior tranches got hurt. What exactly happened?
15. What is the main objective of the Basel III accord?
16. The Basel accords have no legal authority. Why do most developed countries implement them?
17. How is correlation risk related to market risk and to credit risk?
18. How is correlation risk related to systemic risk and to concentration risk?

19. How can we measure the joint probability of occurrence of a binomial event as default or no default?
20. Can it be that two binomial events are negatively correlated but they have a positive probability of joint default?
21. What is value at risk (VaR) and credit value at risk (CVaR)? How are they related?
22. Correlation risk is quite broadly defined in trading practice, referring to any comovement of assets in time. How is the term *correlation* defined in statistics?
23. What do the terms *measure of association* and *measure of dependence* refer to in statistics?

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