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Introduction and Overview

1.1 WHAT IS FIXED INCOME ANALYSIS?

This book develops and studies techniques and models that are helpful in the analysis of fixed income securities. It is difficult to give a clear-cut and universally accepted definition of the term 'fixed income security'. Certainly, the class of fixed income securities includes securities where the issuer promises one or several fixed, predetermined payments at given points in time. This is the case for standard deposit arrangements and bonds. However, we will also consider several related securities as being fixed income securities, although the payoffs of such a security are typically not fixed and known at the time when the investor purchases the security, but depend on the future development in some particular interest rate or the price of some basic fixed income security. In this broader sense of the term, the many different interest rate and bond derivatives are also considered fixed income securities, for example options and futures on bonds or interest rates, caps and floors, swaps and swaptions.

The prices of many fixed income securities are often expressed in terms of various interest rates and yields, so understanding fixed income pricing is equivalent to understanding interest rate behaviour. The key concept in the analysis of fixed income securities and interest rate behaviour is really **the term structure of interest rates**. The interest rate on a loan will normally depend on the maturity of the loan, and in the bond markets there will often be differences between the yields on short-term bonds and long-term bonds. Loosely, the term structure of interest rates is defined as the dependence between interest rates and Maturities. We will be more concrete later on.

We split the overall analysis into two parts which are clearly related to each other. The first part focuses on the economics of the term structure of interest rates, in the sense that the aim is to explore the relations between interest rates and other macroeconomic variables such as aggregate consumption, production, inflation, and money supply. This will help us understand the level of bond prices and interest rates and the shape of the term structure of interest rates at a given point in time and it will give us some tools for understanding and studying the reactions of interest rates and prices to macroeconomic news and shocks. The second part of the analysis focuses on developing tools and models for the pricing and risk management of the many different fixed income securities. Such models are used in all modern financial institutions that trade fixed income securities or are otherwise concerned with the dynamics of interest rates.

In this chapter we will first introduce some basic concepts and terminology and discuss how the term structure of interest rates can be represented in various equivalent ways. In Section 1.3 we take a closer look at the bond and money markets across the world. Among other things we will discuss the size of different markets, the distinction between domestic and international markets, and who the issuers of bonds are. Section 1.4 briefly introduces some fixed income derivatives. Finally, a detailed outline of the rest of the book is given in Section 1.5.

1.2 BASIC BOND MARKET TERMINOLOGY

The simplest fixed income securities are bonds. A bond is nothing but a tradable loan agreement. The issuer sells a contract promising the holder a predetermined payment schedule. Bonds are issued by governments, private and public corporations, and financial institutions. Most bonds are subsequently traded at organized exchanges or over-the-counter (OTC). Bond investors include pension funds and other financial institutions, the central banks, corporations, and households. Bonds are traded with various maturities and with various types of payment schedule. In the so-called money markets major financial institutions offer various bond-like loan agreements of a maturity of less than one year. Below, we will introduce some basic concepts and terminology.

1.2.1 Bond types

We distinguish between zero-coupon bonds and coupon bonds. A **zero-coupon bond** is the simplest possible bond. It promises a single payment at a single future date, the maturity date of the bond. Bonds which promise more than one payment when issued are referred to as **coupon bonds**. We will assume throughout that the face value of any bond is equal to 1 unit of account unless stated otherwise. For concreteness, we will often refer to the unit of account as a 'dollar'. Suppose that at some date t a zero-coupon bond with maturity $T \geq t$ is traded in the financial markets at a price of B_t^T. This price reflects the market **discount factor** for sure time T payments. If many zero-coupon bonds with different maturities are traded, we can form the function $T \mapsto B_t^T$, which we call the market **discount function** prevailing at time t. Note that $B_t^t = 1$, since the value of getting 1 dollar right away is 1 dollar, of course. Presumably, all investors will prefer getting 1 dollar at some time T rather than at a later time S. Therefore, the discount function should be decreasing, that is

$$1 \geq B_t^T \geq B_t^S \geq 0, \quad t \leq T \leq S.$$

A coupon bond has multiple payment dates, which we will generally denote by $T_1, T_2, \dots, T_n$. Without loss of generality we assume that $T_1 < T_2 < \dots < T_n$. The payment at date T_i is denoted by Y_i. For almost all traded coupon bonds the payments occur at regular intervals so that, for all i, $T_{i+1} - T_i = \delta$ for some fixed δ. If we measure time in years, typical bonds have $\delta \in \{0.25, 0.5, 1\}$

corresponding to quarterly, semi-annual, or annual payments. The size of each of the payments is determined by the face value, the coupon rate, and the amortization principle of the bond. The face value is also known as the par value or principal of the bond, and the coupon rate is also called the nominal rate or stated interest rate. In many cases, the coupon rate is quoted as an annual rate even when payments occur more frequently. If a bond with a payment frequency of δ has a quoted coupon rate of R, this means that the periodic coupon rate is δR.

Most coupon bonds are so-called **bullet bonds** or **straight-coupon bonds** where all the payments before the final payment are equal to the product of the coupon rate and the face value. The final payment at the maturity date is the sum of the same interest rate payment and the face value. If R denotes the periodic coupon rate, the payments per unit of face value are therefore

$$Y_i = \begin{cases} R, & i = 1, \dots, n-1 \\ 1 + R, & i = n \end{cases}$$

Of course for $R = 0$ we are back to the zero-coupon bond.

Other bonds are so-called **annuity bonds**, which are constructed so that the total payment is equal for all payment dates. Each payment is the sum of an interest payment and a partial repayment of the face value. The outstanding debt and the interest payment are gradually decreasing over the life of an annuity, so that the repayment increases over time. Let R again denote the periodic coupon rate. Assuming a face value of 1, the constant periodic payment is

$$Y_i = Y \equiv \frac{R}{1 - (1 + R)^{-n}}, \quad i = 1, \dots, n.$$

The outstanding debt of the annuity immediately after the i'th payment is

$$D_i = Y \frac{1 - (1 + R)^{-(n-i)}}{R},$$

the interest part of the i'th payment is

$$I_i = R D_{i-1} = R \frac{1 - (1 + R)^{-(n-i+1)}}{1 - (1 + R)^{-n}},$$

and the repayment part of the i'th payment is

$$X_i = Y (1 + R)^{-(n-i+1)}$$

so that $X_i + I_i = Y$.

Some bonds are so-called **serial bonds** where the face value is paid back in equal instalments. The payment at a given payment date is then the sum of the instalment and the interest rate on the outstanding debt. The interest rate payments, and hence the total payments, will therefore decrease over the life of the bond. With a face value of 1, each instalment or repayment is $X_i = 1/n$, $i = 1, \dots, n$. Immediately after the i'th payment date, the outstanding debt must be $D_i = (n - i)/n = 1 - (i/n)$. The interest payment at T_i is therefore $I_i = R D_{i-1} = R\,(1 - (i - 1)/n)$. Consequently, the total payment at T_i must be

$$Y_i = X_i + I_i = \frac{1}{n} + R\left(1 - \frac{i-1}{n}\right).$$

Finally, few bonds are **perpetuities** or **consols** that last forever and only pay interest, that is $Y_i = R$, $i = 1, 2, \dots$. The face value of a perpetuity is never repaid.

Most coupon bonds have a fixed coupon rate, but a small minority of bonds have coupon rates that are reset periodically over the life of the bond. Such bonds are called *floating rate bonds*. Typically, the coupon rate effective for the payment at the end of one period is set at the beginning of the period at the current market interest rate for that period, for example to the 6-month interest rate for a floating rate bond with semi-annual payments. We will look more closely at the valuation of floating rate bonds in Section 1.2.5.

A coupon bond can be seen as a portfolio of zero-coupon bonds, namely a portfolio of Y_1 zero-coupon bonds maturing at T_1, Y_2 zero-coupon bonds maturing at T_2, and so on. If all these zero-coupon bonds are traded in the market, the price of the coupon bond at any time t must be

$$B_t = \sum_{T_i > t} Y_i B_t^{T_i}, \tag{1.1}$$

where the sum is over all future payment dates of the coupon bond. If this relation does not hold, there will be a clear arbitrage opportunity in the market. The absence of arbitrage is a cornerstone of financial asset pricing, since a market in which prices allow for the construction of arbitrage opportunities cannot be in equilibrium. More on arbitrage and asset pricing theory in Chapter 4.

Example 1.1 Consider a bullet bond with a face value of 100, a coupon rate of 7%, annual payments, and exactly 3 years to maturity. Suppose zero-coupon bonds are traded with face values of 1 dollar and time-to-maturity of 1, 2, and 3 years, respectively. Assume that the prices of these zero-coupon bonds are $B_t^{t+1} = 0.94$, $B_t^{t+2} = 0.90$, and $B_t^{t+3} = 0.87$. According to (1.1), the price of the bullet bond must then be

$$B_t = 7 \cdot 0.94 + 7 \cdot 0.90 + 107 \cdot 0.87 = 105.97.$$

If the price is lower than 105.97, risk-free profits can be locked in by buying the bullet bond and selling 7 1-year, 7 2-year, and 107 3-year zero-coupon bonds. If the price of the bullet bond is higher than 105.97, sell the bullet bond and buy 7 1-year, 7 2-year, and 107 3-year zero-coupon bonds.

If not all the relevant zero-coupon bonds are traded, we cannot justify the relation (1.1) as a result of the no-arbitrage principle. Still, it is a valuable relation. Suppose that an investor has determined (from private or macroeconomic information) a discount function showing the value *she* attributes to payments at different future points in time. Then she can value all sure cash flows in a consistent way by substituting that discount function into (1.1).

The market prices of all bonds reflect a market discount function, which is the result of the supply of and demand for the bonds of all market participants. We can think of the market discount function as a very complex average of the individual discount functions of the market participants. In most markets only few zero-coupon bonds are traded, so that information about the discount function must be inferred from market prices of coupon bonds. We discuss ways of doing that in Chapter 2.

1.2.2 Bond yields and zero-coupon rates

Although discount factors provide full information about how to discount amounts back and forth, it is pretty hard to relate to a 5-year discount factor of 0.7835. It is far easier to relate to the information that the 5-year interest rate is 5%. Interest rates are always quoted on an annual basis, that is as some percentage per year. However, to apply and assess the magnitude of an interest rate, we also need to know the compounding frequency of that rate. More frequent compounding of a given interest rate per year results in higher 'effective' interest rates. Furthermore, we need to know at which time the interest rate is set or observed and for which period of time the interest rate applies. First we consider spot rates which apply to a period beginning at the time the rate is set. In the next subsection, we consider forward rates which apply to a future period of time.

The **yield** of a bond is the discount rate which has the property that the present value of the future payments discounted at that rate is equal to the current price of the bond. The convention in many bond markets is to quote rates using annual compounding. For a coupon bond with current price B_t and payments $Y_1, \dots, Y_n$ at time $T_1, \dots, T_n$, respectively, the annually compounded yield is then the number $\hat{y}_t^B$ satisfying the equation

$$B_t = \sum_{T_i > t} Y_i \left(1 + \hat{y}_t^B\right)^{-(T_i - t)}.$$

Note that the same discount rate is applied to all payments. In particular, for a zero-coupon bond with a payment of 1 at time T, the annually compounded yield $\hat{y}_t^T$ at time $t \leq T$ is such that

$$B_t^T = (1 + \hat{y}_t^T)^{-(T-t)}$$

and, consequently,

$$\hat{y}_t^T = \left(B_t^T\right)^{-1/(T-t)} - 1.$$

We call $\hat{y}_t^T$ the time t **zero-coupon yield**, **zero-coupon rate**, or **spot rate** for date T. The zero-coupon rates as a function of maturity is called the **zero-coupon yield curve** or simply the **yield curve**. It is one way to express the term structure of interest rates. Due to the one-to-one relationship between zero-coupon bond prices and zero-coupon rates, the discount function $T \mapsto B_t^T$ and the zero-coupon yield curve $T \mapsto \hat{y}_t^T$ carry exactly the same information.

Figure 1.1 shows yields on U.S. government bonds of maturities of 1, 5, and 10 years over the period from January 1954 to February 2010. Note the high vari-

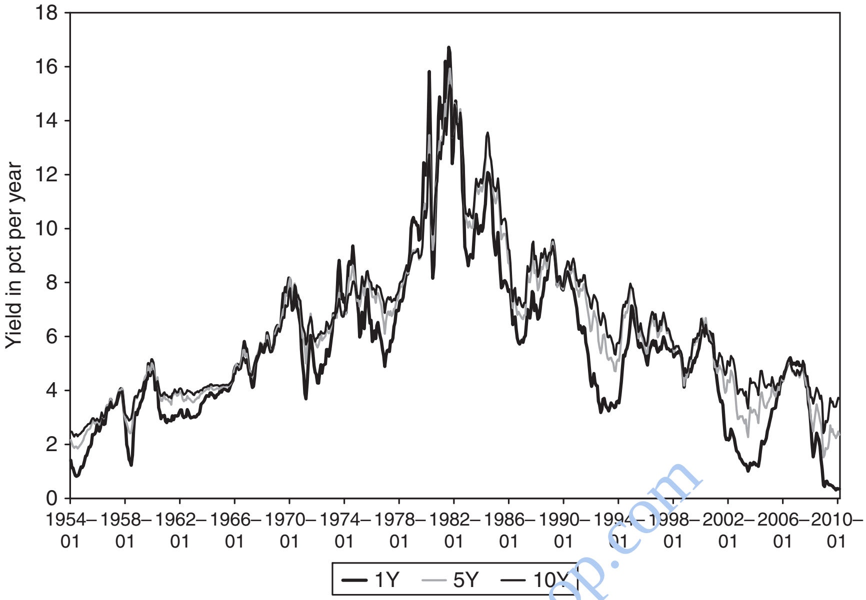

Fig. 1.1: Yields of nominal U.S. Treasury bonds of maturities of 1 year, 5 years, and 10 years from January 1954 to February 2010.

Source: the homepage of the Federal Reserve (`www.federalreserve.gov`) at 2 March, 2010.

ability of the level of interest rates over the full period, whereas over shorter periods yields are quite persistent. Short-maturity yields are more volatile than long-maturity yields. Most of the time, the 1-year yield is less than the 5-year yield which again is less than the 10-year yield indicating that the yield curve is typically upward-sloping. This is also reflected by Fig. 1.2, which shows U.S. yield curves from January in the years 1985, 1990, 1995, 2000, 2005, and 2010. The yield curve from January 1990 is almost flat, but the other yield curves are upward-sloping at least up to maturities of 5–10 years after which the curves are relatively flat. However, in some rather short time periods, short-maturity yields have been higher than long-maturity yields indicating a downward-sloping or *inverted* yield curve. Sometimes the yield curve is non-monotonic and may exhibit a 'hump' (first increasing to a maximum, then decreasing) or a 'trough' (first decreasing to a minimum, then increasing) or have some even more complex shape.

For some bonds and loans interest rates are quoted using semi-annual, quarterly, or monthly compounding. An interest rate of R per year compounded m times a year, corresponds to a discount factor of $(1 + R/m)^{-m}$ over a year. The annually compounded interest rate that corresponds to an interest rate of R compounded m times a year is $(1 + R/m)^m - 1$. This is sometimes called the "effective" interest rate corresponding to the nominal interest rate R. This convention is typically applied for interest rates set for loans at the international money markets, the most commonly used being the LIBOR (London InterBank Offered Rate) rates that are fixed in London. The compounding period equals the maturity of the loan with

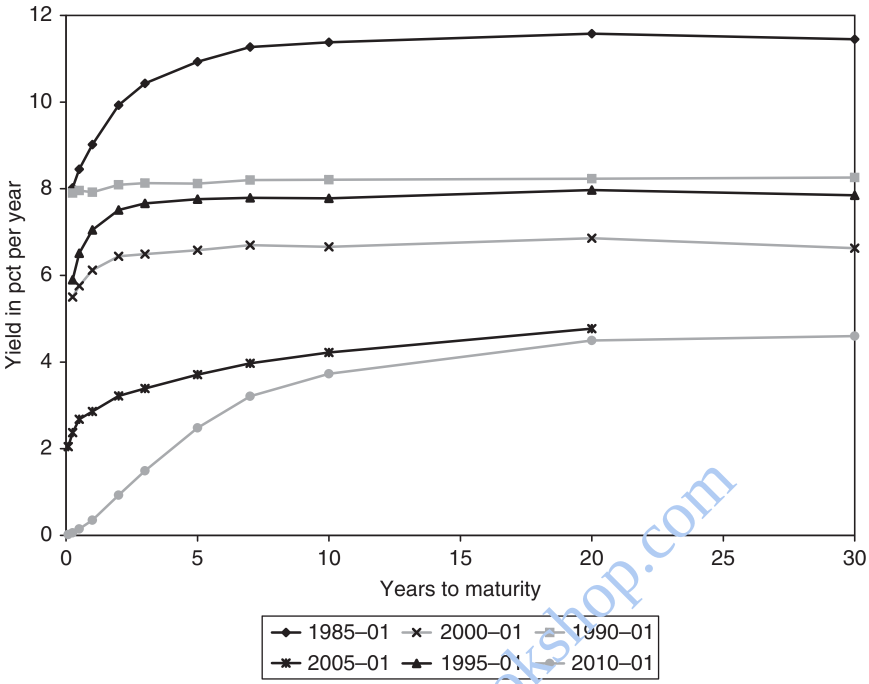

Fig. 1.2: Yield curves determined from nominal US Treasury bonds in January of 1985, 1990, 1995, 2000, 2005, and 2010. The curves are drawn by connecting reported yields for maturities of 1, 3, and 6 months as well as 1, 2, 3, 5, 7, 10, 20, and 30 years. For the curves from 1985, 1990, 1995, and 2000 there is no data for the 1-month yield. For the 1990 curve the 20-year yield is obtained by interpolating the reported 10-year and 30-year yields. For the 2005 curve there is no data for the 30-year yield.

Source: the homepage of the Federal Reserve (`www.federalreserve.gov`) at 2 March, 2010.

3, 6, or 12 months as the most frequently used maturities. If the quoted annualized rate for say a 3-month loan is $l_t^{t+0.25}$, it means that the 3-month interest rate is $l_t^{t+0.25}/4 = 0.25 l_t^{t+0.25}$ so that the present value of 1 dollar paid 3 months from now is

$$B_t^{t+0.25} = \frac{1}{1 + 0.25\, l_t^{t+0.25}}.$$

Hence, the 3-month rate is

$$l_t^{t+0.25} = \frac{1}{0.25}\left(\frac{1}{B_t^{t+0.25}} - 1\right).$$

More generally, the relations are

$$B_t^T = \frac{1}{1 + l_t^T (T - t)} \tag{1.2}$$

and

$$l_t^T = \frac{1}{T-t}\left(\frac{1}{B_t^T} - 1\right).$$

We shall use the term **LIBOR rates** for interest rates that are quoted in this way. Note that if we had a full LIBOR rate curve $T \mapsto l_t^T$, this would carry exactly the same definition as the discount function $T \mapsto B_t^T$. Some fixed income securities provide payoffs that depend on future values of LIBOR rates. In order to price such securities it is natural to model the dynamics of LIBOR rates and this is exactly what is done in one popular class of term structure models (see Chapter 11).

Increasing the compounding frequency m, the effective annual return of 1 dollar invested at the interest rate R per year increases to e^R, due to the mathematical result saying that

$$\lim_{m\to\infty}\left(1 + \frac{R}{m}\right)^m = e^R.$$

A nominal, continuously compounded interest rate R is equivalent to an annually compounded interest rate of $e^R - 1$ (which is bigger than R). Similarly, the zero-coupon bond price B_t^T is related to the continuously compounded zero-coupon rate y_t^T by

$$B_t^T = e^{-y_t^T(T-t)} \tag{1.3}$$

so that

$$y_t^T = -\frac{1}{T-t}\ln B_t^T.$$

The function $T \mapsto y_t^T$ is also a zero-coupon yield curve that contains exactly the same information as the discount function $T \mapsto B_t^T$ and also the same information as the annually compounded yield curve $T \mapsto \hat{y}_t^T$ (or the yield curve with any other compounding frequency). We have the following relation between the continuously compounded and the annually compounded zero-coupon rates:

$$y_t^T = \ln(1 + \hat{y}_t^T).$$

For mathematical convenience we will focus on the continuously compounded yields in most models.

1.2.3 Forward rates

While a zero-coupon or spot rate reflects the price on a loan between today and a given future date, a **forward rate** reflects the price on a loan between two future dates. The annually compounded relevant forward rate at time t for the period between time T and time S is denoted by $\hat{f}_t^{T,S}$. Here, we have $t \le T < S$. This is the rate which is appropriate at time t for discounting between time T and S. We can think of discounting from time S back to time t by first discounting from time S to time T and then discounting from time T to time t. We must therefore have that

$$\left(1+\hat{y}_t^S\right)^{-(S-t)} = \left(1+\hat{y}_t^T\right)^{-(T-t)} \left(1+\hat{f}_t^{T,S}\right)^{-(S-T)}, \tag{1.4}$$

from which we find that

$$\hat{f}_t^{T,S} = \frac{(1+\hat{y}_t^T)^{-(T-t)/(S-T)}}{(1+\hat{y}_t^S)^{-(S-t)/(S-T)}} - 1.$$

We can also write (1.4) in terms of zero-coupon bond prices as

$$B_t^S = B_t^T \left(1+\hat{f}_t^{T,S}\right)^{-(S-T)}, \tag{1.5}$$

so that the forward rate is given by

$$\hat{f}_t^{T,S} = \left(\frac{B_t^T}{B_t^S}\right)^{1/(S-T)} - 1.$$

Note that since $B_t^t = 1$, we have

$$\hat{f}_t^{t,S} = \left(\frac{B_t^t}{B_t^S}\right)^{1/(S-t)} - 1 = \left(B_t^S\right)^{-1/(S-t)} - 1 = \hat{y}_t^S,$$

that is the forward rate for a period starting today equals the zero-coupon rate or spot rate for the same period.

Again, we may use periodic compounding. For example, a 6-month forward LIBOR rate of $L_t^{T,T+0.5}$ valid for the period $[T, T+0.5]$ means that the discount factor is

$$B_t^{T+0.5} = B_t^T \left(1 + 0.5\, L_t^{T,T+0.5}\right)^{-1}$$

so that

$$L_t^{T,T+0.5} = \frac{1}{0.5}\left(\frac{B_t^T}{B_t^{T+0.5}} - 1\right).$$

More generally, the time t forward LIBOR rate for the period $[T, S]$ is given by

$$L_t^{T,S} = \frac{1}{S-T}\left(\frac{B_t^T}{B_t^S} - 1\right). \tag{1.6}$$

If $f_t^{T,S}$ denotes the continuously compounded forward rate prevailing at time t for the period between T and S, we must have that

$$B_t^S = B_t^T e^{-f_t^{T,S}(S-T)},$$

by analogy with (1.5). Consequently,

$$f_t^{T,S} = -\frac{\ln B_t^S - \ln B_t^T}{S-T}. \tag{1.7}$$

Using (1.3), we get the following relation between zero-coupon rates and forward rates under continuous compounding:

$$f_t^{T,S} = \frac{y_t^S(S-t) - y_t^T(T-t)}{S-T}. \tag{1.8}$$

In the following chapters, we shall often focus on forward rates for future periods of infinitesimal length. The forward rate for an infinitesimal period starting at time T is simply referred to as the forward rate for time T and is defined as $f_t^T = \lim_{S\to T} f_t^{T,S}$. The function $T \mapsto f_t^T$ is called the **term structure of forward rates** or the **forward rate curve**. Letting $S \to T$ in the expression (1.7), we get

$$f_t^T = -\frac{\partial \ln B_t^T}{\partial T} = -\frac{\partial B_t^T/\partial T}{B_t^T}, \tag{1.9}$$

assuming that the discount function $T \mapsto B_t^T$ is differentiable. Conversely,

$$B_t^T = e^{-\int_t^T f_t^u\,du}. \tag{1.10}$$

Note that a full term structure of forward rates $T \mapsto f_t^T$ contains the same information as the discount function $T \mapsto B_t^T$.

Applying (1.8), the relation between the infinitesimal forward rate and the spot rates can be written as

$$f_t^T = \frac{\partial[y_t^T(T-t)]}{\partial T} = y_t^T + \frac{\partial y_t^T}{\partial T}(T-t)$$

under the assumption of a differentiable term structure of spot rates $T \mapsto y_t^T$. The forward rate reflects the slope of the zero-coupon yield curve. In particular, the forward rate f_t^T and the zero-coupon rate y_t^T will coincide if and only if the zero-coupon yield curve has a horizontal tangent at T. Conversely, we see from (1.10) and (1.3) that

$$y_t^T = \frac{1}{T-t}\int_t^T f_t^u\,du, \tag{1.11}$$

that is the zero-coupon rate is an average of the forward rates.

1.2.4 The term structure of interest rates in different disguises

We emphasize that discount factors, spot rates, and forward rates (with any compounding frequency) are perfectly equivalent ways of expressing the same information. If a complete yield curve of, say, quarterly compounded spot rates is given, we can compute the discount function and spot rates and forward rates for any given period and with any given compounding frequency. If a complete term structure of forward rates is known, we can compute discount functions and spot rates, and so on. Academics frequently apply continuous compounding since the mathematics involved in many relevant computations is more elegant when exponentials are used, but continuously compounded rates can easily be transformed to any other compounding frequency.

There are even more ways of representing the term structure of interest rates. Since most bonds are bullet bonds, many traders and analysts are used to thinking in terms of yields of bullet bonds rather than in terms of discount factors or zero-coupon rates. The **par yield** for a given maturity is the coupon rate that causes a bullet bond of the given maturity to have a price equal to its face value. Again we have to fix the coupon period of the bond. U.S. treasury bonds typically have semi-annual coupons which are therefore often used when computing par yields. Given a discount function $T \mapsto B_t^T$, the n-year par yield is the value of c that solves the equation

$$\sum_{i=1}^{2n} \left(\frac{c}{2}\right) B_t^{t+0.5i} + B_t^{t+n} = 1.$$

It reflects the current market interest rate for an n-year bullet bond. The par yield is closely related to the so-called swap rate, which is a key concept in the swap markets, compare Section 6.5.

1.2.5 Floating rate bonds

Floating rate bonds have coupon rates that are reset periodically over the life of the bond. We will consider the most common floating rate bond, which is a bullet bond, where the coupon rate effective for the payment at the end of one period is set at the beginning of the period at the current market interest rate for that period.

Assume again that the payment dates of the bond are $T_1 < \cdots < T_n$, where $T_i - T_{i-1} = \delta$ for all i. The annualized coupon rate valid for the period $[T_{i-1}, T_i]$ is the δ-period market rate at date T_{i-1} computed with a compounding frequency of δ. We will denote this interest rate by $l_{T_{i-1}}^{T_i}$, although the rate is not necessarily a LIBOR rate, but can also be a Treasury rate. If the face value of the bond is H, the payment at time T_i $(i = 1, 2, \dots, n-1)$ equals $H\delta l_{T_{i-1}}^{T_i}$, while the final payment at time T_n equals $H(1 + \delta l_{T_{n-1}}^{T_n})$. If we define $T_0 = T_1 - \delta$, the dates $T_0, T_1, \dots, T_{n-1}$ are often referred to as the reset dates of the bond.

We will argue that immediately after each reset date, the value of the bond will equal its face value. To see this, first note that immediately after the last reset date T_{n-1}, the bond is equivalent to a zero-coupon bond with a coupon rate equal to the market interest rate for the last coupon period. By definition of that market interest rate, the time T_{n-1} value of the bond will be exactly equal to the face value H. In mathematical terms, the market discount factor to apply for the discounting of time T_n payments back to time T_{n-1} is $(1 + \delta l_{T_{n-1}}^{T_n})^{-1}$, so the time T_{n-1} value of a payment of $H(1 + \delta l_{T_{n-1}}^{T_n})$ at time T_n is precisely H. Immediately after the next-to-last reset date T_{n-2}, we know that we will receive a payment of $H\delta l_{T_{n-2}}^{T_{n-1}}$ at time T_{n-1} and that the time T_{n-1} value of the following payment (received at T_n) equals H. We therefore have to discount the sum $H\delta l_{T_{n-2}}^{T_{n-1}} + H = H(1 + \delta l_{T_{n-2}}^{T_{n-1}})$ from T_{n-1} back to T_{n-2}. The discounted value is exactly H. Continuing this procedure, we get that immediately after a reset of the coupon rate, the floating rate bond is valued at par. Note that it is crucial for this result that the coupon rate is adjusted to the interest rate considered by the market to be 'fair'.

We can also derive the value of the floating rate bond between two payment dates. Suppose we are interested in the value at some time t between T_0 and T_n. Introduce the notation

$$i(t) = \min \{i \in \{1, 2, \dots, n\} : T_i > t\}, \tag{1.12}$$

so that $T_{i(t)}$ is the nearest following payment date after time t. We know that the following payment at time $T_{i(t)}$ equals $H\delta l_{T_{i(t)-1}}^{T_{i(t)}}$ and that the value at time $T_{i(t)}$ of all the remaining payments will equal H. The value of the bond at time t will then be

$$B_t^{\mathrm{fl}} = H(1 + \delta l_{T_{i(t)-1}}^{T_{i(t)}}) B_t^{T_{i(t)}}, \quad T_0 \le t < T_n. \tag{1.13}$$

This expression also holds at payment dates $t = T_i$, where it results in H, which is the value excluding the payment at that date.

Relatively few floating rate bonds are traded, but the results above are also very useful for the analysis of interest rate swaps studied in Section 6.5.

1.3 BOND MARKETS AND MONEY MARKETS

This section will give an overview of the bond and money markets across the world. We can distinguish between national markets and international markets. In the national market of a country, bonds primarily issued by domestic issuers and aimed at domestic investors are traded, although some bonds issued by certain foreign governments or corporations or international associations are often also traded. The bonds issued in a given national market must comply with the regulation of that particular country. The international bond market is often referred to as the Eurobond market. A Eurobond can have an issuer located in one country, be listed on the exchange in a second country, and be denominated in the currency of a third country. Eurobonds are usually underwritten by an international syndicate and offered to investors in several countries simultaneously. The Eurobond market is less regulated than most national markets. Eurobonds are typically listed on one national exchange, but most of the trading in these bonds takes place in rather well-organized OTC markets. Other Eurobonds are issued as a private placement with financial institutions. Eurobonds are typically issued by international institutions, governments, or large multi-national corporations.

The Bank for International Settlements (BIS) regularly publishes statistics on financial markets across the world. BIS distinguishes between domestic debt and international debt securities. The term 'debt securities' covers both bonds and money market contracts. The term 'domestic' means that the security is issued in the local currency by residents in that country and targeted at resident investors. All other debt securities are classified by BIS as 'international'. Based on BIS statistics published in Bank for International Settlements (2010), henceforth referred to as BIS (2010), Table 1.1 ranks domestic markets for debt securities according to the amounts outstanding in September 2009. The United States has by far the largest domestic bond market, while Japan is a clear number two. The size of the domestic bond market relative to GDP varies significantly across countries. For example, both Belgium and Denmark have larger domestic bond markets relative to GDP, whereas China and the United Kingdom have smaller bond markets when compared to their GDP. In several countries, including the U.S., the domestic bond market has a size similar to the stock market.

Table 1.2 lists the countries most active when it comes to issuing international debt securities. The domestic markets are significantly larger than the international markets, and international bond markets are much larger than international money markets. Relative to their GDP, European countries such as Germany, the United Kingdom, and the Netherlands have a large share of the international bond and money markets, whereas U.S.-based issuers are relatively inactive. This is also reflected by Table 1.3 which shows that the Euro is the most frequently used currency in the international markets for debt securities, but the U.S. dollar is also used very often.

Table 1.1: The largest domestic markets for debt securities divided by issuer category as of September 2009.

Country	Amounts outstanding (billion USD)	Fraction of world market (%)	Fraction of domestic market (%)		
			governments	financial institut.	corporate issuers
United States	25,105	39.0	36.5	52.3	11.2
Japan	11,602	18.0	83.6	9.6	6.8
Italy	3,770	5.8	54.5	32.3	13.2
France	3,189	4.9	53.1	38.1	8.9
Germany	2,927	4.5	53.3	34.9	11.8
China	2,413	3.7	58.7	28.9	12.4
Spain	2,071	3.2	34.0	30.7	35.3
United Kingdom	1,566	2.4	72.9	25.7	1.4
Canada	1,260	2.0	68.7	20.6	10.7
Brazil	1,227	1.9	65.3	34.0	0.7
South Korea	1,071	1.7	39.7	29.9	30.4
The Netherlands	1,003	1.6	38.4	51.5	10.1
Australia	843	1.3	24.7	70.9	4.4
Belgium	724	1.1	59.9	31.2	8.9
Denmark	589	0.9	16.7	83.0	0.3
All countries	64,448	100.0	52.7	36.4	10.9

Source: Tables 16A-B in BIS (2010).

Table 1.2: International debt securities by nationality of issuer as of December 2009. The numbers are amounts outstanding in billions of USD.

Country	Total	Security (%)		Issuer (%)		
		bonds, notes	money market	governments	financial institut.	corporate issuers
United States	6,712	99.0	1.0	0.2	81.5	18.4
United Kingdom	3,174	96.3	3.7	1.6	89.2	9.2
Germany	2,932	96.1	3.9	10.3	85.2	4.5
France	2,017	95.0	5.0	2.8	77.4	19.9
Spain	1,824	95.1	4.9	7.8	89.1	3.1
Italy	1,402	96.9	3.1	17.5	75.8	6.7
The Netherlands	1,285	93.2	6.8	1.8	92.8	5.4
Ireland	589	91.4	8.6	10.6	87.7	1.7
Belgium	586	95.3	4.7	25.0	69.4	5.6
Canada	569	99.0	1.0	17.6	55.7	26.7
Intl. organizations	802	99.0	1.0	NA	NA	NA
All countries	27,010	96.5	3.5	8.4	77.3	11.3

Source: Tables 12A-D and 15A-B in BIS (2010).

Table 1.3: International debt securities by currency. The numbers are amounts outstanding in billions of USD as of December 2009.

Currency	Bonds and notes	Money market
Euro	12,386	443
US dollar	9,429	320
Pound sterling	2,149	99
Yen	691	17
Swiss franc	366	21
Canadian dollar	306	1
Australian dollar	268	10
Swedish krona	69	2
Hong Kong dollar	61	9
Norwegian krone	54	1
Other currencies	300	10
Total	26,078	932

Source: Tables 13A-B in BIS (2010).

Tables 1.1 and 1.2 split up the different markets according to three categories of issuers: governments, financial institutions, and corporate issuers. On average, close to 53% of the debt securities traded in domestic markets are issued by governments, 36% by financial institutions, and 11% by corporate issuers. Note the large difference across countries. Some domestic markets (for example Japan and the United Kingdom) are dominated by government bonds, others (for example Denmark and Australia) by bonds issued by financial institutions, and corporate bonds are also very common in some countries (for example Spain and South Korea) while virtually non-existent in other countries. The international markets are dominated by financial institutions who stand behind approximately 77% of the issues, 11% are issued by corporations, 8% by governments, and 3% by international organizations. Some governments (for example Belgium, Canada, and Italy) often issue bonds on the international market, while others (for example the United States and the United Kingdom) rarely do so. Let us look more closely at the different issuers and the type of debt securities they typically issue.

Government bonds are bonds issued by the government to finance and refinance the public debt. In most countries, such bonds can be considered to be free of default risk, and interest rates in the government bond market are then a benchmark against which the interest rates on other bonds are measured. However, in some economically or politically unstable countries, the default risk on government bonds cannot be ignored.[1] In the U.S., government bonds are issued by the Department of the Treasury and are referred to as Treasury securities or just Treasuries. These securities are divided into three categories: bills, notes, and bonds. **Treasury bills** (or simply T-bills) are short-term securities that mature in one year or less from their issue date. T-bills are zero-coupon bonds since they have a single payment equal to the face value. **Treasury notes and bonds** are coupon-bearing bullet bonds with semi-annual payments. The only difference between notes and bonds is the time-to-maturity when first issued. Treasury notes are issued with a time-to-maturity of 1–10 years, while Treasury bonds mature in more than 10 years and up to 30 years from their issue date. The Treasury sells two types of notes and bonds: fixed-principal and inflation-indexed. The fixed-principal type promises given dollar payments in the future, whereas the dollar payments of the inflation-indexed type are adjusted to reflect inflation in consumer prices.[2] Finally, the U.S. Treasury also issue so-called **savings bonds** to individuals and certain organizations, but these bonds are not subsequently tradable.

While Treasury notes and bonds are issued as coupon bonds, the Treasury Department introduced the so-called STRIPS program in 1985 that lets investors hold and trade the individual interest and principal components of most Treasury notes and bonds as separate securities.[3] These separate securities, which are usually referred to as STRIPs, are zero-coupon bonds. Market participants create STRIPs by separating the interest and principal parts of a Treasury note or bond. For example, a 10-year Treasury note consists of 20 semi-annual interest payments and a principal payment payable at maturity. When this security is 'stripped', each of the 20 interest payments and the principal payment become separate securities and can be held and transferred separately.

In some countries, including the U.S., bonds issued by various public institutions, for example utility companies, railway companies, export support funds, and so on are backed by the government, so that the default risk on such bonds is the risk that the government defaults. In addition, some bonds are issued by government-sponsored entities created to facilitate borrowing and reduce bor-

[1] This risk is not hypothetical. Tomz and Wright (2007) report that 106 countries have defaulted on their debt a total of 250 times in the period 1820–2004. For more on default risk, see Chapter 13.

[2] The principal value of an inflation-indexed note or bond is adjusted before each payment date according to the change in the consumer price index. Since the semi-annual interest payments are computed as the product of the fixed coupon rate and the current principal, all the payments of an inflation-indexed note or bond are inflation-adjusted.

[3] STRIPS is short for Separate Trading of Registered Interest and Principal of Securities.

rowing costs for, for example farmers, homeowners, and students. However, these bonds are typically not backed by the government and are therefore exposed to the risk of default of the issuing organization. Bonds may also be issued by local governments. In the U.S. such bonds are known as municipal bonds.

In the United States and some other countries, corporations will traditionally raise large amounts of capital by issuing bonds, so-called **corporate bonds**. In other countries, for example Germany and Japan, corporations borrow funds primarily through bank loans, so that the market for corporate bonds is limited. For corporate bonds, investors cannot ignore the possibility that the issuer defaults and cannot meet the obligations represented by the bonds. Bond investors can either perform their own analysis of the creditworthiness of the issuer or rely on the analysis of professional rating agencies such as Moody's Investors Service or Standard & Poor's Corporation. These agencies designate letter codes to bond issuers both in the U.S. and in other countries. Investors will typically treat bonds with the same rating as having (nearly) the same default risk. Due to the default risk, corporate bonds are traded at lower prices than similar (default-free) government bonds. The management of the issuing corporation can effectively transfer wealth from bond-holders to equity-holders, for example by increasing dividends, taking on more risky investment projects, or issuing new bonds with the same or even higher priority in case of default. Corporate bonds are often issued with bond covenants or bond indentures that restrict management from implementing such actions. Default risk, credit ratings, and the valuation of corporate bonds will be thoroughly discussed in Chapter 13.

U.S. corporate bonds are typically issued with maturities of 10–30 years and are often callable bonds, so that the issuer has the right to buy back the bonds on certain terms (at given points in time and for a given price). Some corporate bonds are convertible bonds meaning that the bond-holders may convert the bonds into stocks of the issuing corporation on predetermined terms. Although most corporate bonds are listed on a national exchange, much of the trading in these bonds is in the OTC market.

When commercial banks and other financial institutions issue bonds, the promised payments are sometimes linked to the payments on a pool of loans that the issuing institution has provided to households or firms. An important example is the class of **mortgage-backed bonds** which constitutes a large part of some bond markets, for example in the U.S., Germany, Denmark, Sweden, and Switzerland. A mortgage is a loan that can (partly) finance the borrower's purchase of a given real estate property, which is then used as collateral for the loan. Mortgages can be residential (family houses, apartments, and so on) or non-residential (corporations, farms, and so on). The issuer of the loan (the lender) is a financial institution. Typical mortgages have a maturity between 15 and 30 years and are annuities in the sense that the total scheduled payment (interest plus repayment) at all payment dates are identical. Fixed-rate mortgages have a fixed interest rate, while adjustable-rate mortgages have an interest rate which is reset periodically according to some reference rate. A characteristic feature of most mortgages is the prepayment option. At any payment date in the life of the loan, the borrower has the right to pay off all or part of the outstanding debt. This can occur due to a sale of the underlying real estate property, but can also occur after a drop in market interest rates, since the borrower then has the chance to get a cheaper loan.

Mortgages are pooled either by the issuers or other institutions, who then issue mortgage-backed securities that have an ownership interest in a given pool of mortgage loans. The most common type of mortgage-backed securities is the so-called **pass-through**, where the pooling institution simply collects the payments from borrowers with loans in a given pool and 'passes through' the cash flow to investors less some servicing and guaranteeing fees. Many pass-throughs have payment schemes equal to the payment schemes of bonds, for example pass-throughs issued on the basis of a pool of fixed-rate annuity mortgage loans have a payment schedule equal to that of annuity bonds. However, when borrowers in the pool prepay their mortgage, these prepayments are also passed through to the security-holders, so that their payments will be different from annuities. In general, owners of pass-through securities must take into account the risk that the mortgage borrowers in the pool default on their loans. In the U.S. most pass-throughs have been issued by three organizations that guarantee the payments to the securities even if borrowers default. These organizations are the Government National Mortgage Association (called 'Ginnie Mae'), the Federal Home Loan Mortgage Corporation ('Freddie Mac'), and the Federal National Mortgage Association ('Fannie Mae'). The securities issued by these institutions have generally been considered free of default risk. There is more on mortgages and mortgage-backed securities in Chapter 14.

The **money market** is a market for borrowing and lending large amounts over a period of up to one year. The major players in the money market are financial institutions and large private corporations. The debt contracts issued in the money market are mainly zero-coupon loans, which have a single repayment date. The loans are implemented by the issuance of various instruments. Large corporations, both financial corporations and others, often finance short-term liquidity needs by issuing so-called **commercial paper**. Another standard money market contract is a **repurchase agreement** or simply **repo**. One party of this contract sells a certain asset, such as a short-term Treasury bill, to the other party and promises to buy back that asset at a given future date at the market price at that date. A repo is effectively a collateralized loan, where the underlying asset serves as collateral. As central banks in other countries, the Federal Reserve in the U.S. participates actively in the repo market to implement its monetary policy. The interest rate on repos is called the repo rate. Other popular instruments in the money market are certificates of deposit and foreign exchange swaps, but the money markets also include standard deposits, forward rate agreements, and trading in Treasury bills and short-lived asset-backed securities.

The central bank is a key player in the money market. Banks keep deposits in the central bank to comply with reserve requirements, to facilitate financial transactions, and to manage short-term liquidity. If one bank is in need of additional reserves, it can borrow money (usually overnight) from another bank with surplus reserves at the central bank. The interest rate on the loan is negotiated between the two banks. In the U.S., the weighted average of these interest rates across all such transactions is referred to as the federal funds (or just Fed funds) effective rate. This rate is a crucial determinant of the interest rates offered and charged by banks to their customers. The Fed funds rate is also very important for floating rate bonds and the prices and yields of short-term bonds. The governors of the Federal Reserve set and regularly reconsider a target Fed funds rate based on their view of the

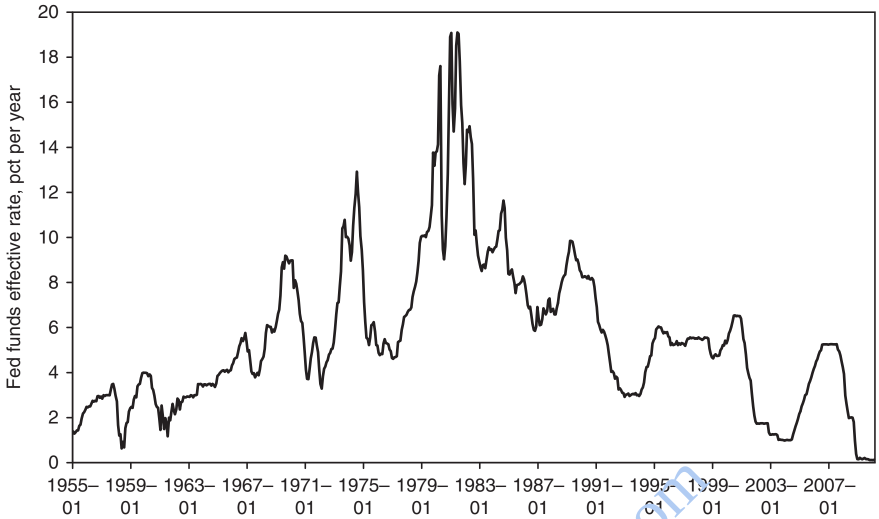

Fig. 1.3: The Fed funds effective rate from July 1954 to February 2010.

Source: the homepage of the Federal Reserve (www.federalreserve.gov) at 2 March, 2010.

current and future economic conditions. The Federal Reserve then buys and sells securities in open market operations to manage the liquidity in the market, thereby also affecting the Fed funds effective rate. Figure 1.3 depicts the Fed funds effective rate in the period from July 1954 to February 2010. Banks may obtain temporary credit directly from the Federal Reserve at the so-called 'discount window'. The interest rate charged by the Fed on such credit is called the federal discount rate, but since such borrowing is quite uncommon nowadays, the federal discount rate serves more as a signaling device for the targets of the Federal Reserve.

Many of the contracts used in the money markets are benchmarked to (that is priced by reference to) the London Interbank Offered Rate (LIBOR) for the appropriate term and currency. In the Euromarket, deposits are negotiated for various terms and currencies, but most deposits are in U.S. dollars or the Euro for a period of 1, 3, or 6 months. Interest rates set on unsecured deposits at the London interbank market are called LIBOR rates.

More details on U.S. bond markets can be found in Fabozzi (2010), while Batten et al. (2004) provide detailed information on European bond and money markets.

1.4 FIXED INCOME DERIVATIVES

A wide variety of fixed income derivatives are traded around the world. In this section we provide a brief introduction to the markets for such securities. In the pricing models we develop in later chapters we will look for prices of some of the most popular fixed income derivatives. Chapter 6 contains more details on a number of fixed income derivatives, what cash flow they offer, how the different derivatives are related, and so on. Credit-related derivatives are discussed in Chapter 13, where key statistics for those markets are also provided.

A **forward** is the simplest derivative. A forward contract is an agreement between two parties on a given transaction at a given future point in time and at a price that is already fixed when the agreement is made. For example, a forward on a bond is a contract where the parties agree to trade a given bond at a future point in time for a price which is already fixed today. This fixed price is usually set so that the value of the contract at the time of inception is equal to zero. In that case no money changes hands before the delivery date. A closely related contract is the so-called **forward rate agreement (FRA)**. Here the two parties agree that one party will borrow money from the other party over some period beginning at a given future date and the interest rate for that loan is fixed already when this FRA is entered. In other words, the interest rate for the future period is locked in. FRAs are quite popular instruments in the money markets.

As a forward contract, a **futures** contract is an agreement upon a specified future transaction, for example a trade of a given security. The special feature of a future is that changes in its value are settled continuously throughout the life of the contract (usually once every trading day). This so-called **marking-to-market** ensures that the value of the contract (that is the value of the future payments) is zero immediately following a settlement. This procedure makes it practically possible to trade futures at organized exchanges, since there is no need to keep track of when the futures position was originally taken. Futures on government bonds are traded at many leading exchanges. A very popular exchange-traded derivative is the so-called **Eurodollar futures**, which is basically the futures equivalent of a forward rate agreement.

An **option** gives the holder the right to make some specified future transaction at terms that are already fixed. A call option gives the holder the right to buy a given security at a given price at or before a given date. Conversely, a put option gives the holder the right to sell a given security. If the option gives the right to make the transaction at only one given date, the option is said to be European-style. If the right can be exercised at any point in time up to some given date, the option is said to be American-style. Both European- and American-style options are traded. Options on government bonds are traded at several exchanges and also on the OTC-markets. In addition, many bonds are issued with 'embedded' options. For example, many mortgage-backed bonds and corporate bonds are callable, in the sense that the issuer has the right to buy back the bond at a pre-specified price. To value such bonds, we must be able to value the option element.

Various interest rate options are also traded in the fixed income markets. The most popular are **caps and floors**. A cap is designed to protect an investor who has borrowed funds on a floating interest rate basis against the risk of paying very high interest rates. Therefore the cap basically gives you the right to borrow at some given rate. A cap can be seen as a portfolio of interest rate call options. Conversely, a floor is designed to protect an investor who has lent funds on a floating rate basis against receiving very low interest rates. A floor is a portfolio of interest rate put options. Various exotic versions of caps and floors are also quite popular.

A **swap** is an exchange of two cash flow streams that are determined by certain interest rates. In the simplest and most common interest rate swap, a **plain vanilla** swap, two parties exchange a stream of fixed interest rate payments and a stream

Table 1.4: Derivatives traded on organized exchanges. All amounts are in billions of U.S. dollars. The amount outstanding is of December 2009, whereas the turnover figures are for the fourth quarter of 2009.

Instruments/ Location	Futures		Options	
	Amount outstanding	Turnover	Amount outstanding	Turnover
All markets	21,749	307,315	51,388	137,182
Interest rate	20,623	276,215	46,435	106,523
Currency	164	7,677	147	582
Equity index	962	23,423	4,807	30,077
North America	10,716	156,160	23,875	55,216
Europe	8,054	129,016	26,331	62,937
Asia-Pacific	2,446	18,567	310	17,235
Other markets	532	3,573	873	1,793

Source: Table 23A in BIS (2010).

of floating interest rate payments. There are also currency swaps where streams of payments in different currencies are exchanged. In addition, many exotic swaps with special features are widely used. The international OTC swap markets are huge, both in terms of transactions and outstanding contracts. The **credit default swap** or just CDS is a widely used swap contract, in which the buyer makes a series of payments to the seller and, in exchange, receives a payoff if there is a default (or another 'credit event') on a certain bond or loan issued by a third party. More on credit default swaps and other credit-related securities can be found in Chapter 13.

A **swaption** is an option on a swap, that is it gives the holder the right, but not the obligation, to enter into a specific swap with pre-specified terms at or before a given future date. Both European- and American-style swaptions are traded.

The Bank for International Settlements (BIS) also publishes statistics on derivative trading around the world. Table 1.4 provides some interesting statistics on the size of derivatives markets at organized exchanges. The markets for interest rate derivatives are much larger than the markets for currency- or equity-linked derivatives. The option markets generally dominate futures markets measured by the amounts outstanding, but ranked according to turnover futures markets are larger than options markets.

The BIS statistics also contain information about the size of OTC markets for derivatives. BIS estimates that in June 2009 the total amount outstanding on OTC derivative markets was 604,622 billion of U.S. dollars, of which single-currency interest rate derivatives account for 72.3%, currency derivatives account for 8.1%, credit default swaps for 6.0%, equity-linked derivatives for 1.1%, commodity contracts for 0.6%, while the remainder cannot be split into any of these categories, compare Table 19 in BIS (2010). Table 1.5 shows how the interest rate derivatives market can be disaggregated according to instrument and maturity. Approximately 36.7% of these OTC-traded interest rate derivatives are denominated in Euros, 35.3% in U.S. dollars, 13.1% in yen, 7.5% in pounds sterling, and the remaining 7.4% in other currencies, compare Table 21B in BIS (2010).

Table 1.5: Amounts outstanding (billions of U.S. dollars) on OTC single-currency interest rate derivatives as of June 2009.

<table>
<tr><th rowspan="2">Contracts</th><th rowspan="2">Total</th><th colspan="3">Maturity in years</th></tr>
<tr><th>≤1</th><th>1–5</th><th>≥5</th></tr>
<tr><td>All interest rates</td><td>437,198</td><td>159,143</td><td>128,301</td><td>149,754</td></tr>
<tr><td>Forward rate agreements</td><td>46,798</td><td rowspan="2">150,630</td><td rowspan="2">111,431</td><td rowspan="2">126,623</td></tr>
<tr><td>Swaps</td><td>341,886</td></tr>
<tr><td>Options</td><td>48,513</td><td>8,513</td><td>16,870</td><td>23,130</td></tr>
</table>

Source: Tables 21A and 21C in BIS (2010).

1.5 AN OVERVIEW OF THE BOOK

The key element in our analysis will be the term structure of interest rates. The cleanest picture of the link between interest rates and maturities is given by a zero-coupon yield curve. In many markets only a few zero-coupon bonds are traded, so that we have to extract an estimate of the zero-coupon yield curve from prices of the traded coupon bonds. We will discuss methods for doing that in Chapter 2.

Risk is a central concept in fixed income modelling. The prices and yields of bonds are affected by the expectation and uncertainty about future values of certain macroeconomic variables. The prices of fixed income derivatives reflect the present value of the future payoff, which is generally dependent on the value of some bond price or interest rate at a later date. Consequently, we need to model the behaviour of uncertain variables or objects over time. This is done in terms of stochastic processes. A stochastic process is basically a collection of random variables, namely one random variable for each of the points in time at which we are interested in the value of this object. To understand and work with modern fixed income models therefore requires some knowledge about stochastic processes, their properties, and how to do relevant calculations involving stochastic processes. Chapter 3 provides the information about stochastic processes that is needed for our purposes.

This book focuses on the pricing of fixed income securities. However, the pricing of fixed income securities follows the same general principles as the pricing of all other financial assets. Chapter 4 reviews some of the important results on asset pricing theory. In particular, we define and relate the key concepts of arbitrage, state-price deflators, and risk-neutral probability measures. The connections to market completeness and individual investors' behaviour are also addressed. All these results will be applied in the following chapters to the term structure of interest rates and the pricing of fixed income securities.

In Chapter 5 we study the links between the term structure of interest rates and macroeconomic variables such as aggregate consumption, production, and inflation. The term structure of interest rates reflects the prices of bonds of various maturities and, as always, prices are set to align supply and demand. An investor with a clear preference for current capital to finance investments or current consumption can borrow by issuing a bond to an investor with a clear preference for future consumption opportunities. The price of a bond of a given maturity will therefore depend on the attractiveness of the real investment opportunities and on the individuals' preferences for consumption over the maturity of the bond. Following this intuition we develop relations between interest rates, aggregate consumption, and aggregate production. We also explore the relations between nominal interest rates, real interest rates, and inflation. Finally, the chapter reviews some of the traditional hypotheses on the shape of the yield curve, for example the expectation hypotheses, and discuss their relevance (or, rather, irrelevance) for modern fixed income analysis.

Chapter 6 provides an overview of the most popular fixed income derivatives, such as futures and options on bonds, Eurodollar futures, caps and floors, and swaps and swaptions. We will look at the characteristics of these securities and what we can say about their prices without setting up any concrete term structure model.

Starting with Chapter 7 we focus on dynamic term structure models developed for the pricing of fixed income securities and the management of interest rate risk. Chapter 7 goes through so-called one-factor diffusion models. This type of model was the first to be applied in the literature and dates back at least to 1970. The one-factor models of Vasicek and Cox, Ingersoll, and Ross are still frequently applied both in practice and in academic research. They have a lot of realistic features and deliver simple pricing formulas for many fixed income securities. Chapter 8 explores multi-factor diffusion models which have several advantages over one-factor models, but are also more complicated to analyse and apply.

The diffusion models deliver prices both for bonds and derivatives. However, the model price for a given bond may not be identical to the actually observed price of the bond. If you want to price a derivative on that bond, this seems problematic. If the model does not get the price of the underlying security right, why trust the model's price of the derivative? In Chapter 9 we illustrate how one-factor diffusion models can be extended to be consistent with current market information, such as bond prices and volatilities. A more direct route to ensuring consistency is explored in Chapter 10 that introduces and analyses so-called Heath–Jarrow–Morton models. They are characterized by taking the current market term structure of interest rates as given and then modelling the evolution of the entire term structure in an arbitrage-free way. We will explore the relation between these models and the factor models studied in earlier chapters.

Yet another class of models is the subject of Chapter 11. These 'market models' are designed for the pricing and hedging of specific products that are traded on a large scale in the international markets, namely caps, floors, and swaptions. These models have become increasingly popular in recent years.

In Chapters 6–11 we focus on the pricing of various fixed income securities. However, it is also extremely important to be able to measure and manage interest rate risk. Interest rate risk measures of individual securities are needed in order to obtain an overview of the total interest rate risk of the investors' portfolio and to identify the contribution of each security to this total risk. Many institutional investors are required to produce such risk measures for regulatory authorities and for publication in their accounting reports. In addition, such risk measures constitute an important input to the portfolio management. Interest rate risk management is the topic of Chapter 12. First, some traditional interest rate risk measures are reviewed and criticized. Then we turn to risk measures defined in relation to the dynamic term structure models studied in the previous chapters.

The following chapters deal with some securities that require special attention. In Chapter 13 we discuss the pricing of corporate bonds and other fixed income securities where the default risk of the issuer cannot be ignored. The subject of Chapter 14 is how to construct models for the pricing and risk management of mortgage-backed securities. The main concern is how to adjust the models studied in earlier chapters to take the prepayment options involved in mortgages into account. Chapter 15 focuses on the consequences that stochastic variations in interest rates have for the valuation of securities with payments that are not directly related to interest rates, such as stock options and currency options.

Finally, Chapter 16 describes and illustrates several numerical techniques that are often applied in cases where explicit pricing and hedging formulas are not available.

1.6 EXERCISES

Exercise 1.1 Show that if the discount function does *not* satisfy the condition

$$B_t^T \geq B_t^S, \quad t \leq T < S,$$

then negative forward rates will exist. Are non-negative forward rates likely to exist? Explain!

Exercise 1.2 Consider two bullet bonds, both with annual payments and exactly 4 years to maturity. The first bond has a coupon rate of 6% and is traded at a price of 101.00. The other bond has a coupon rate of 4% and is traded at a price of 93.20. What is the 4-year discount factor? What is the 4-year zero-coupon interest rate?

Exercise 1.3 Consider a bond market in which the annually compounded zero-coupon yields of maturities from 1 to 5 years are

$$\hat{y}^1 = 5\%, \quad \hat{y}^2 = 6\%, \quad \hat{y}^3 = 6.8\%, \quad \hat{y}^4 = 7.4\%, \quad \hat{y}^5 = 7.5\%.$$

What are the corresponding discount factors B^T, $T = 1, 2, \dots, 5$? What are the 1-year forward rates $\hat{f}^{T,T+1}$, $T = 0, 1, \dots, 4$?

Exercise 1.4 Consider a coupon bond with payments Y_i at time $T_i = i$, $i = 1, \dots, n$, such that is there is a payment of Y_1 in 1 year, a payment of Y_2 in 2 years, and so on. Suppose you discount all future payments with a constant, annually compounded interest rate of r. Let B denote the present value, that is

$$B = \sum_{i=1}^{n} Y_i (1+r)^{-i}.$$

(a) Show that if the bond is a *bullet bond* with a coupon rate of R and a face value of 1, then

$$B = \frac{R}{r} + \left(1 - \frac{R}{r}\right)(1+r)^{-n}.$$

(b) Show that if the bond is an *annuity bond* with a coupon rate of R and a face value of 1, then

$$B = \frac{\alpha(n,r)}{\alpha(n,R)},$$

where $\alpha(N,\rho) = \rho^{-1}\left(1-(1+\rho)^{-N}\right)$.

(c) Show that if the bond is a *serial bond* with a coupon rate of R and a face value of 1, then

$$B = \frac{R}{r} + \frac{1}{n}\left(1-\frac{R}{r}\right)\alpha(n,r).$$

Exercise 1.5 What bonds are currently traded in your domestic market? Try to find information about historical interest rates in your country, either yields on government bonds or official interest rates fixed by the central bank or both.